Dividing a Commons under Tight Guarantees

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Abstract

We revisit the "self-ownership" viewpoint to regulate the utilisation of common property resources. Allocating to each agent "the fruit of their own labor" is typically ill-defined, so we look for tight approximations of this decentralised ideal. For each agent i two guarantees limit, from above and below, the impact of other agents on i's allocation. They limit the range of unscripted negotiations, or the choice of a full sharing rule.

Our context-free model of the commons is a mapping $\mathcal{W}$ from profiles of "types" to a freely transferable amount of benefit or cost. If $\mathcal{W}$ is super (resp. sub) modular there is a single tight upper (resp. lower) guarantee, and an infinite menu of tight lower (resp. upper) guarantees, each one conveying a precise normative viewpoint. We describe the menu for essentially all modular two person problems, and familiar examples like the allocation of an indivisible item, cooperative production, and facility location.

JEL classification codes: D6, D63

1 Introduction

The efficient exploitation of common property resources, that we call a commons for short, includes the division of private commodities, sharing a production function, assigning costs to jointly liable agents etc... Even under the strong simplifying assumptions that the agents have identical rights to these resources and are fully responsible for their own type (which may represent preferences, needs, skills, efforts, location ...) it is not clear how to take efficiently into account the differences in individual types while fairly respecting the equality of the agents' rights?

A first uncontroversial step is Horizontal Equity: two agents with identical types must be treated equally, a compelling property when, as they do here, types capture all the features relevant to the allocation problem under scrutiny. We assume the stronger property called Anonymity*:* swapping the (possibly different) types of two agents exchanges their allocations and does not affect that of other agents.

These two properties as well as the Efficiency requirement, are *context-free* (independent of the physical description of the commons), universally applicable, and always bite.

We propose a new context-free fairness principle to manage a commons, inspired by the Lockean maxim that each agent should receive "the fruit of their own labor" ([16]), in modern terminology

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the "self-ownership" viewpoint (more on this in the literature review). In our abstract model of the commons where agents are responsible for their own type, the proposal to "reward each agent for their own type" is appealing but hopelessly ambiguous: there is typically no clear way to disentangle the contributions of individual types to a particular outcome.

Consider for instance the classic cooperative production model where n agents each enter an input x_i (their type) in the function F and must share the output $F(x_N)$ (where $x_N = \sum_{i=1}^n x_i$). Individual contributions to the output are unambiguous if, and only if, F has constant returns to scale (CRS): $F(x_N) = \sum_{i=1}^n F(x_i)$. Then the division of the output can be decentralised as if each agent was using their input in a private copy of the technology and i's share is $F(x_i)$.[1]

In our abstract context-free commons the resources are black-boxed as a function $\mathcal{W}$ inputting a n-profile x of types x_i and returning the output $\mathcal{W}(x)$ (benefit or cost). For tractability we assume that output is freely transferable accross agents, for instance by cash transfers. Moreover $\mathcal{W}$ is symmetric in its n variables, which is necessary to make sense of the Anonymity property. The allocation of output is *decentralisable* if the function $\mathcal{W}$ is separably additive: for some real valued function g of a single type we have

$$\mathcal{W}(x) = \sum_{i=1}^{n} g(x_i)$$

for all profiles $x = (x_i)_1^n$.

Taking this decentralisation property as normatively desirable but out of reach in all interesting commons problems, we approximate it. If my share cannot be independent of other agents' types, the second best goal is to minimise this influence as much as possible.

Formally we look for a pair of *guarantees* g^- and g^+ such that, for all profiles $x = (x_i)_1^n$,

$$\sum_{1}^{n} g^-(x_i) \leq \mathcal{W}(x) \leq \sum_{1}^{n} g^+(x_i) \tag{1}$$

and in addition these guarantees are "tight": increasing the lower guarantee $g^-(x_i)$ (resp. decreasing the upper guarantee $g^+(x_i)$) at any type x_i violates the left hand (LH) (resp. the RH) inequality above at some profile containing x_i.

In the next four paragraphs we briefly describe our general results (Sections 4 and 7) about the tight solutions of system (1) for an abstract function $\mathcal{W}$. The proof that our approach casts a useful light on the perennial challenge of managing a commons is in the pudding of a half-dozen concrete microeconomic examples developed in Sections 2, 5 and 6.

The first clue about the set of tight guarantees is a simple observation. Define the *unanimity share* of type x_i as his fair share at the hypothetical profile denoted $(\overset{n}{x_i})$ where all agents have the same type: $una(x_i) = \frac{1}{n}\mathcal{W}(\overset{n}{x_i})$. At such unanimous profile the inequalities imply, whether $g^{\pm}$ are tight or not,

$$g^-(x_i) \leq una(x_i) \leq g^+(x_i)$$

If the function $x_i \to una(x_i)$ itself is a lower guarantee it must be tight,[2] therefore it is the *only* tight lower guarantee. Insisting that type x_i's share of output be at least $una(x_i)$ is the reasonable statement that differences in types should be to everyone's (weak) advantage. Similarly if una is an upper guarantee it is the only tight one, and we require that differences in types be to everyone's (weak) disadvantage.[3]

[1] This solution of the CRS commons is often taken as a primitive requirement in the axiomatic discussion of the cooperative production problem (e. g. [21]); it can also be deduced from its incentive properties ([15]).

[2] If $una(x_i) < g^-(x_i)$ then g^- does not produce a feasible lower guarantee at $(\overset{n}{x_i})$.

[3] These properties are called "diversity of preferences dividend" and "of preferences burden" in ([32]), p.112-114.

The proof of the second key fact requires more work. If the function $\mathcal{W}$ describing the commons is supermodular[4] then *una* is an upper guarantee, hence the unique tight one; and if $\mathcal{W}$ is submodular *una* is the unique tight lower guarantee (Proposition 4.1). Most of our results and examples apply to modular functions to take advantage of this crucial simplification.

On the other side of the tight unanimity bound we find instead an infinite menu of tight guarantees. Each one of these places different limits on individual claims or liabilities and suggests a different interpretation of individual rights as a function of responsible types. Each example, starting in Secion 2, adds a context to the abstract commons and normative content to the menu of tight guarantees.

In Section 4 we also identify, for any modular function $\mathcal{W}$, two tight guarantees on the other side of the unanimity one. These two *incremental* guarantees denoted g_{inc}, g^{inc} adapt to our model the *stand alone* share of an agent using the commons without sharing it with anyone else (Proposition 4.2); if H and L are respectively the largest and smallest type, they achieve the largest and the smallest gap $g(H) - g(L)$ among all tight guarantees (Lemma 4.1).

The next general result, Theorem 7.1, is a full characterisation of all tight lower (resp. upper) guarantees for two person commons with one-dimensional types, when $\mathcal{W}(x_1, x_2)$ is strictly super- or submodular. They are parametrised by the choice of a decreasing, continuous and symmetric function from the set of types into itself. Therefore their set is of infinite dimension, and the same is true with more than two agents.

Section 5 focuses on the classic commons $\mathcal{W}(x) = F(x_N)$ illustrated when F maps inputs to output (Example 5.1), demands to cost (Section 5.3), or location on a line to transportation costs (Example 5.2). The unanimity share $una(x_i) = \frac{1}{n}F(nx_i)$ is the tight lower guarantee if F is convex ($\mathcal{W}$ supermodular), the tight upper one iff F is concave ($\mathcal{W}$ submodular). Although we do not describe the full set of tight guarantees we identify two interesting subsets, both of them linking the two incremental guarantees common to all modular functions. The first one has only $(n-2)$ guarantees of the stand alone type: Proposition 5.1. The second set is a continuous line, containing most of the tangents to the graph of the unanimity function: Proposition 5.2.

Our second characterisation result applies to a rich class of functions $\mathcal{W}$ modular but not strictly so (unlike Theorem 7.1) where the types x_i are again one dimensional. The first example, in the introductory Section 2, is the submodular commons $\mathcal{W}(x) = \max_i\{x_i\}$ that we interpret in two ways: sharing the cost of a public facility ([14]) or assigning an indivisible item and cash transfers. The tight lower guarantee is $una(x_i) = \frac{1}{n}x_i$; on the other side there is a one dimensional set of tight upper guarantees parametrised by a "benchmark" type p and easy to interpret: Proposition 2.1.

Theorem 6.1 is a considerable generalisation of this first example. Write the order statistics of $(x_i)_1^n$ as $(x^k)_1^n$ (where $x^1 = \max_i\{x_i\}$ and $x^n = \min_i\{x_i\}$) and call *rank separable* a commons of the form $\mathcal{W}(x) = \sum_{[n]} w_k(x^k)$. If such a function is modular (requiring the $w_{k+1} - w_k$ to be increasing in type) then the tight guarantees (opposite to the the unanimity one) are parametrised by an arbitrary $(n-1)$-vector or types $c = (c_k)_1^{n-1}$ in the following stand alone form:

$$g^-(x_i) = \mathcal{W}(x_i; c) - \frac{1}{n}(\sum_{k=1}^{n-1} \mathcal{W}(c_k; c))$$

Applications include capacity and facility location problems: Examples 6.1, 6.2. We also describe the tight guarantees of the *non modular* commons $\mathcal{W}(x) = x^k$ for k between 2 and $n-1$: Example 6.3.

[4]If $x_1 < x_1^*$ the difference $\mathcal{W}(x_1^*, x_2, x_{-1,2}) - \mathcal{W}(x_1, x_2, x_{-1,2})$ increases weakly in x_2.

tight guarantees and sharing rules Tight guarantees help to manage a commons in two ways.

First to promote participation in an unscripted negotiation by minimising its stakes: a type x_i agent can reject any agreement where her share falls outside the interval $[g^-(x_i), g^+(x_i)]$, confident that if no agreement is reached the manager will pick an (arbitrary) outcome within these bounds for each i.

Second, it delivers a sharp normative test for any deterministic division rule, that is any mapping φ computing at each profile of types x the individual shares $(\varphi_i(x))_1^n$ such that $\mathcal{W}(x) = \sum_{i-1}^n \varphi_i(x)$. Each such rule implements its own lower and upper guarantees for each type x_i: $g^-(x_i) = \min_{x_{-i}}\{\varphi_i(x_i, x_{-i})\}$ and $g^+(x_i) = \max_{x_{-i}}\{\varphi_i(x_i, x_{-i})\}$. Our test dismisses the rules implementing non tight guarantees and partitions the others according to the tight pair they generate; each set of this infinite partition is itself infinite (Lemma 3.3).

In the classic commons $\mathcal{W}(x) = F(x_N)$ with F concave it is easy to see that the venerable Average Return (AR) rule $\varphi_i(x) = \frac{x_i}{x_N} F(x_N)$ gives to each type x_i below the average type a share below the unanimity lower guarantee $\frac{1}{n}F(nx_i)$, imposing on them low returns that they are not responsible for ([21]).[5] Lemma 5.1 dismisses similarly the Shapley sharing rule.

But the increasing and decreasing Serial rules (Definition 4.3) implement respectively the tight pairs (una, g_{inc}) and (una, g^{inc}). This statement generalises to all modular functions $\mathcal{W}$: Proposition 4.3.

It may not be easy, given a particular sharing rule, to decide if it implements tight guarantees or not. But the answer to the converse question is easy: given a pair of tight guarantees any sharing rule delivering shares within the interval it defines implements exactly these guarantees (Lemma 3.3). We can construct such rules by simple extrapolations of the guarantees, or by "trimming" an arbitrary sharing rule when its shares violate the guarantees.

related literature The first modern mathematical fair division model, cast in the context of cake cutting ([29], [12] [9]), had a simple message:if utilities are non atomic and additive over the cake every agent can guarantee a fair $\frac{1}{n}$ share of her utility for the whole cake. This is precisely the unanimity guarantee, after generalising our abstract model without transferable utility.

When economists joined the discussion of fair division in the early 70-s, the concept of *endogenous fair share* – in our terminology a guaranteed worst case utility against other adversarial participants – maintained its prominence. If we divide a bundle ω of private Arrow Debreu (AD) commodities and preferences are convex, the allocation$\frac{1}{n}\omega$ is the compelling fair share because it delivers the unanimity utility ([34], [31]); the latter is also recognised in the allocation of indivisible items with cash transfers ([30], [2]).

In the next two decades, endogenous lower and upper bounds on individual welfare play an important role in the axiomatic discussion of cooperative production. The stand alone utility (from using a private copy of the production function) joins the unanimity utility and sits on the opposite side of the Pareto frontier when the returns to scale are monotonic ([27], [18], [21], [36]). The same is true in the public good provision model irrespective of returns ([19]); see also a proposal of weaker bounds in ([10]).

Endogenous guarantees appear also in the axiomatic bargaining model ([33]) with a focus on variations in the set of agents rather than the agents' preferences.

In this century computer scientists and others are still searching for a compelling concept of fair share for the allocation of indivisible items (good or bad) even when utilities are additive: only

[5] If F is convex AR allows agents with a below average input x_i to free ride on high returns they did not contribute to, contradicting again the unanimity upper guarantee.

a little more than a $\frac{3}{4}$ fraction of the plausible *MaxMinShare* (MMS) is feasible at all profiles ([6], [25], [1]) and this may or may not define a tight guarantee; see also other concepts in ([8]), ([17]), ([13]) and ([3]).

Another widely open question is the search for tight guarantees in cake division with non additive utilities or for dividing AD commodities with non convex preferences ([4]).

Our interpretation of self-ownership as a pair of lower and upper bounds on welfare introduces the new viewpoint that minimising the best case utility (against adversarial others) is as important as maximising the worst case utility. The assumption that the agents are fully responsible for their own type avoids a familiar controversy around the "neo-Lockean" self-ownership maxim, defended by Nozick ([24]) but criticised for its potentially libertarian implications by Roemer ([26]) and Cohen ([7]).

We also diverge from the context dependent interpretation of common ownership derived from the Resource Monotonicity property: when the shared resources improve, all the participants should weakly benefit. This fairness test is in fact incompatible with the unanimity fair share if we divide AD commodities ([23]), and with the unanimity utility in the cooperative production commons ([22]).

Our abstract model of the commons delivers general results (in Sections 3, 4) that we apply to a great variety of microeconomic examples. But the assumption that utility is transferable by some numeraire is typically absent in the above literature.

contents of the next sections An asterisk signals those already described above.

Section 2$^\bigstar$: the introductory example.

Section 3 introduces the model for a general function $\mathcal{W}$ and domain $\mathcal{X}$, lists various topological properties, and two critical Lipschitz and differentiability properties that tight guarantees inherit from the function $\mathcal{W}$. The long list of technical Lemmas can be skipped by the reader impatient to discover the implications of tight guarantees in the concrete commons of sections 4, 5 and 6.

Section 4$^\bigstar$: general results for modular functions: on one side *una* is the unique tight guarantee, on the other two canonical incremental tight guarantees.

Section 5$^\bigstar$: the classic commons $\mathcal{W}(x) = F(x_N)$.

Section 6$^\bigstar$: the modular rank separable functions and their tight guarantees (Theorem 6.1).

Section 7$^\bigstar$: the tight guarantees of strictly modular two person commons (Theorem 7.1).

Section 8 collects two open questions and some take home points

Section 9 is an Appendix gathering several long or minor proofs.

2 A canonical example

Example 2.1 *sharing the cost of a capacity ([14])*

The n agents share a public facility (canal, broadband,...) adjusted to their different needs (for a canal more or less wide or deep, for a small or large broadband). The cost of building enough capacity to serve the needs of agent i is x_i; the cost of serving everyone is $\mathcal{W}(x) = \max_{i\in[n]}\{x_i\}$, that must be divided in n shares y_i s. t. $\sum_{[n]} y_i = \max_{i\in[n]}\{x_i\}$. The range of possible individual needs x_i is the interval $[L, H]$ where $0 < L < H$.

The unanimity share $una(x_i) = \frac{1}{n}x_i$ is a lower guarantee because $\sum_{[n]} \frac{1}{n}x_i \leq \max_{i\in[n]}\{x_i\}$, therefore as noted in the Introduction *una* is the *only* tight guarantee on the Left Side of (1): agent i should pay at least her fair share of the capacity she needs. Our first result describes all solutions of the RH inequalities.

We use the notation $z_+ = \max\{z, 0\}$ and the proof uses two easy Lemmas from Section 3.

Proposition 2.1: *The minimal upper guarantees g^+_p of $\mathcal{W}(x) = \max_{1\leq i\leq n}\{x_i\}$ are parametrised by a benchmark type $p \in [L, H]$ as follows:*

$$g^+_p(x_i) = \frac{1}{n}p + (x_i - p)_+ \text{ for } x_i \in [L, H] \tag{2}$$

Proof[6]: Checking that g^+_p meets the RH inequalities in (1) is routine and omitted for brevity.

Next we pick an arbitrary tight upper guarantee g^+ and set $p = ng^+(L)$. At the unanimous profile of L-s the RH of (1) implies $p \geq L$. Tightness implies that g^+ increases weakly (Lemma 3.4) so $g^+(x_i) \geq \frac{1}{n}p$ for all x_i. The constant function $\frac{1}{n}H$ is an upper guarantee therefore if $p > H$ g^+ is not tight: so $p \in [L, H]$.

Inequality (1) applied to x_i and $n-1$ types L gives $g^+(x_i) \geq x_i - \frac{n-1}{n}p$; combining this with $g^+(x_i) \geq \frac{1}{n}p$ gives $g^+ \geq g^+_p$. Because g^+ is tight and g^+_p is an upper guarantee this must be an equality, which also implies that g^+_p is tight. ■

The two end-points guarantees in (2) are

$$g^+_L(x_i) = x_i - \frac{n-1}{n}L \text{ ; } g^+_H(x_i) = \frac{1}{n}H \text{ for } x_i \in [L, H]$$

Type x_i's worst cost share in g^+_L is $\frac{n-1}{n}(x_i - L)$ over and above the best one $una(x_i) = \frac{1}{n}x_i$.[7] In contrast g_H is the same worst cost share for everyone. The pair of guarantees (una, g^+_L) implies that a type L *always pays exactly* $\frac{1}{n}L$, while under (una, g^+_H) she can pay as much as $\frac{1}{n}H$; vice versa a type H *always* pays $\frac{1}{n}H$ under the latter pair and as much as $H - \frac{n-1}{n}L$ under the former. Agents with small needs prefer g^+_L above g^+_H, those with large needs g^+_H above g^+_L.

Under the compromise pair (una, g^+_p) the benchmark type p always pays $\frac{1}{n}p$ and this is the worst cost share for all types below p, similar to g^+_H, while types above p can pay as much as $x_i - \frac{n-1}{n}p$, similar to g^+_L.

In certain profiles of types the pair (una, g^+_p) determines the full profile of shares: say $x_{i^*} \geq p$ and $x_j \leq p$ for all $j \neq i^*$, then $y_{i^*} = x_{i^*} - \frac{n-1}{n}p$ and $y_j = \frac{1}{n}p$ for $j \neq i^*$.

The *serial sharing rule* (proposed by ([14]) for this problem) implements the tight pair (una, g^+_L). Order the agents by increasing type, then charge $y_1 = \frac{1}{n}L + \frac{1}{n}(x_i - L), y_2 = y_1 + \frac{1}{n-1}(x_2 - x_1)$, etc.. The easy proof is a special case of Proposition 4.3. The tight pair (una, g^+_H) is implemented by the much simpler Equal-Split rule $y_i = \frac{1}{n}\max_{i\in[n]}\{x_i\}$, ignoring all differences in types.

To implement the pair (una, g^+_p) we can for instance combine these two rules as follows (there are many other ways: Lemma 3.3). Write $\widetilde{x}_i = \min\{x_i, p\}$, $\widehat{x}_i = (x_i - p)_+$, and define

$$\varphi^p(x) = \varphi^{egal}(\widetilde{x}) + \varphi^{ser}(\widehat{x})$$

where φ^{egal} is Equal-Split and the serial rule φ^{ser} is applied to the interval $[0, H - p]$.

Importantly the familiar Proportional sharing rule $\varphi^{pro}_i(x) = \frac{x_i}{x_N}\max_{j\in[n]}\{x_j\}$ (well defined everywhere because $L > 0$) implements neither a tight lower nor a tight upper guarantee. Its best and worst case cost shares are easily computed as

$$\begin{aligned} g^-_{pro}(x_i) &= \frac{x_i^2}{x_i + (n-1)H} \\ g^+_{pro}(x_i) &= \max\{\frac{x_i^2}{x_i + (n-1)L}, \frac{Hx_i}{x_i + H + (n-2)L}\} \end{aligned} \tag{3}$$

[6] The result is a special case of Theorem 6.1, but this redundant short proof is much easier to follow.

[7] In Section 3.2 g^+_L will be the incremental guarantee starting from L, denoted g_{inc}.

so that $g^-_{pro}(x_i) < \frac{1}{n}x_i$ for all x_i except H, with equality at H.

Clearly g^+_{pro} increases in x_i and $g^+_{pro}(L) = \frac{HL}{H+(n-1)L}$. Set $\overline{p} = \frac{nHL}{H+(n-1)L}$ and mimick the steps in the proof of Proposition 2.1 to conclude that $g^+_{pro}(x_i) \geq g^+_{\overline{p}}(x_i)$ for all x_i. This is an equality at L and H but a strict inequality in the neighborhood of L and H, therefore $g^+_{\overline{p}}$ improves upon g^+_{pro}.

Remark 2.1 *In general the Equal-Split rule is not even compatible with our interpretation of self-ownership for separably additive functions $\mathcal{W}$. The fact that it implements here the pair of tight guarantees (una, g^{inc}) is an interesting exception: this situation essentially characterises this iconic example. Suppose $\mathcal{W}$ is weakly increasing and submodular (defined in Section 4) in $[L,H]^{[n]}$. If the equal split rule $y_i = \frac{1}{n}\mathcal{W}(x)$ implements the unanimity lower guarantee then $\min_{x_{-i}} \frac{1}{n}\mathcal{W}(x_i; x_{-i}) = \frac{1}{n}\mathcal{W}(x_i; \overset{n-1}{L})$ must equal $una(x_i) = \frac{1}{n}\mathcal{W}(\overset{n}{x_i})$ (Proposition 4.1); this implies that $\mathcal{W}$ takes the form $\mathcal{W}(x) = w(\max_{i\in[n]})$ for some weakly increasing function w in $\mathcal{X}$; and Equal-Split also implements the flat upper guarantee $\frac{1}{n}w(H)$.*

3 General model

The set of agents is $[n] = \{1, \cdots, n\}$ and $\mathcal{X}$ is the common set of types. All properties in this section apply if $\mathcal{X}$ is a compact subset of a general euclidian space $\mathbb{R}^A$ partially ordered in the usual way, an assumption maintained in this section except for statement *ii*) in Lemma 3.9 where $\mathcal{X}$ is a compact interval in $\mathbb{R}^A$.

At the profile $x = (x_i)_{i\in[n]} \in \mathcal{X}^{[n]}$ we must divide the benefit or cost $\mathcal{W}(x)$. The function $\mathcal{W}$ is symmetric in the n variables x_i and continuous.

A division of $\mathcal{W}(x)$ is $y = (y_i)_{i\in[n]} \in \mathbb{R}^{[n]}$ s. t. $\sum_{[n]} y_i = \mathcal{W}(x)$. A sharing rule maps each profile $x \in \mathcal{X}^{[n]}$ to a division of $\mathcal{W}(x)$.

3.1 lower and upper guarantees

Definition 3.1: *The functions g^- and g^+ from $\mathcal{X}$ into $\mathbb{R}$ are respectively a lower and an upper guarantee of $\mathcal{W}$ if and only if they satisfy the inequalities:* for $x \in \mathcal{X}^{[n]}$

$$\sum_{i\in[n]} g^-(x_i) \leq \mathcal{W}(x) \leq \sum_{i\in[n]} g^+(x_i) \text{ for } x \in \mathcal{X}^{[n]} \tag{4}$$

We write $\boldsymbol{G}^-, \boldsymbol{G}^+$ the sets of such guarantees.

Given two lower guarantees $g^1, g^2 \in \boldsymbol{G}^-$ we say that g^1 dominates g^2 if $g^1(x_i) \geq g^2(x_i)$ for $x_i \in \mathcal{X}$ and $g^1 \neq g^2$. The guarantee $g \in \boldsymbol{G}^-$ is *tight* if increasing its value at a single $x_1 \in \mathcal{X}$ creates a violation of the LS inequality in (4) for some $x_{-1} \in \mathcal{X}^{[n-1]}$.

The isomorphic statement for upper guarantees in $\boldsymbol{G}^+$ flips the domination inequality around and for tightness replaces increasing by decreasing and LS by RS.

We write $\mathcal{G}^-$ and $\mathcal{G}^+$ for the subsets of tight guarantees in $\boldsymbol{G}^-$ and $\boldsymbol{G}^+$.

Lemma 3.1 *For $\varepsilon = +, -$ every guarantee $g \in \boldsymbol{G}^\varepsilon \diagdown \mathcal{G}^\varepsilon$ is dominated by a tight one.*

This is a simple application of Zorn's Lemma.

The restriction of $\mathcal{W}$ to the diagonal of $\mathcal{X}^{[n]}$ defines the *unanimity* share of agent i:

$$una(x_i) = \frac{1}{n}\mathcal{W}(\overset{n}{x_i}) \tag{5}$$

where the notation $(\overset{m}{z})$ is the m-vector with z in each coordinate.

We repeat for completeness two observations made in the Introduction.

Lemma 3.2
i) For any $(g^-, g^+) \in \boldsymbol{G}^- \times \boldsymbol{G}^+$ *and for* $x_i \in \mathcal{X}$

$$g^-(x_i) \leq una(x_i) \leq g^+(x_i) \tag{6}$$

ii) If una is a lower guarantee it dominates each lower guarantee; this is also true for upper guarantees. For $\varepsilon = +, -$:

$$una \in \boldsymbol{G}^\varepsilon \Longrightarrow \mathcal{G}^\varepsilon = \{una\}$$

If $\mathcal{W}$ is additively separable it takes the form $\mathcal{W}(x) = \sum_{[n]} una(x_i)$ and statement *ii)* implies $\mathcal{G}^\varepsilon = \{una\}$ for $\varepsilon = +, -$. Conversely if $\mathcal{G}^\varepsilon = \{una\}$ for $\varepsilon = +, -$ then una satisfies both sides of (4) so that $\mathcal{W}$ is additively separable. In any other case there is a real choice of at least one type of tight guarantees.

A pair of tight guarantees is implemented by many different sharing rules φ.

Lemma 3 3. *Fix the function* $\mathcal{W}$ *and a tight pair* $(g^-, g^+) \in \mathcal{G}^- \times \mathcal{G}^+$*. If the sharing rule* φ *satisfies* $g^-(x_i) \leq \varphi_i(x) \leq g^+(x_i)$ *for all* i *and* x *then it implements* (g^-, g^+)*:* for all i and x

$$\min_{x_{-i}}\{\varphi_i(x_i, x_{-i})\} = g^-(x_i) \ ; \ \max_{x_{-i}}\{\varphi_i(x_i, x_{-i})\} = g^+(x_i)$$

This follows at once from the tightness of g^- and g^+.

The moving average of g^- and g^+ is the simplest sharing rule implementing this pair in $\mathcal{G}^- \times \mathcal{G}^+$:

$$\varphi_i(x) = \lambda g^-(x_i) + (1 - \lambda) g^+(x_i)$$

where λ is chosen s. t. for all $x \in \mathcal{X}^{[n]}$

$$\lambda \sum_{[n]} g^-(x_i) + (1 - \lambda) \sum_{[n]} g^+(x_i) = \mathcal{W}(x)$$

Also, for any given sharing rule φ that does not implement (g^-, g^+) it is easy to adjust it *only* at those profiles where if fails at least one of these bounds so that the adjusted rule $\widetilde{\varphi}$ does implement the pair of guarantees and preserves the choices of φ as much as possible.

3.2 regularity and topological properties

Lemma 3.4 *If* $\mathcal{X}$ *is ordered by* $\succ$ *and* $\mathcal{W}$ *is weakly increasing in* x*, so is every tight guarantee in* $\mathcal{G}^\varepsilon$, for $\varepsilon = +, -$.

Proof Fix $g \in \mathcal{G}^-$. If $x_i \succ x'_i$ and $g(x_i) < g(x'_i)$ define $\widetilde{g}(x_i) = g(x'_i)$ and $\widetilde{g} = g$ otherwise, then check that $\widetilde{g}$ is still in $\boldsymbol{G}^-$. This contradicts that g is tight. ■

Lemma 3.5 *For* $\varepsilon = +, -$*, fix an equi-continuous function* $\mathcal{W}$ *in* $\mathcal{X}^{[n]}$.
i) A tight guarantee $g \in \mathcal{G}^\varepsilon$ *is continuous in* $\mathcal{X}$*.*
ii) A guarantee g *in* $\boldsymbol{G}^\varepsilon$ *is tight if and only if:* for all $x_i \in \mathcal{X}$ there exists $x_{-i} \in \mathcal{X}^{[n-1]}$ s. t.

$$g(x_i) + \sum_{j \in [n] \diagdown i} g(x_j) = \mathcal{W}(x_i, x_{-i}) \tag{7}$$

Then we call (x_i, x_{-i}) *a contact profile of* g *at* x_i *; the set of such profiles is the contact set* $\mathcal{C}(g)$ *of* g*.*

Proof in the Appendix 9.1.

Lemma 3.6 *For* $\varepsilon = +, -$,
i) For any $x_1 \in \mathcal{X}$ *there is a tight guarantee* $g \in \mathcal{G}^\varepsilon$ *s.t.* $g(x_1) = una(x_1)$.
ii) The set $\mathcal{G}^\varepsilon$ *is a singleton only if it contains una.*
Proof in the Appendix 9.2.

By statement *ii*) and the comments after Lemma 3.2 we see that $\mathcal{G}^-$ and $\mathcal{G}^+$ are both singletons if and only if $\mathcal{W}$ is additively separable.

Next we state without proof two useful invariance properties.

Lemma 3.7 *For* $\varepsilon = +, -$,
i) If $\mathcal{W}_0$ *is additively separable,* $\mathcal{W}_0(x) = \sum_{[n]} w_0(x_i)$, *and* $\mathcal{W}$ *is symmetric on* $\mathcal{X}^{[n]}$ *we have*

$$\mathcal{G}^\varepsilon(\mathcal{W} + \mathcal{W}_0) = \mathcal{G}^\varepsilon(\mathcal{W}) + \{w_0\}$$

ii) Change of the type variable. If θ *is a bicontinuous increasing bijection* $x_i = \theta(z_i)$ *from* $\mathcal{Z}$ *into* $\mathcal{X}$, $\mathcal{W}$ *is defined on* $\mathcal{X}^{[n]}$ *and* $g \in \mathcal{G}^\varepsilon(\mathcal{W})$, *then* $g \circ \theta \in \mathcal{G}^\varepsilon(\widetilde{\mathcal{W}})$ *where* $\widetilde{\mathcal{W}}(z) = \mathcal{W}(\theta(z))$ *and* $\theta(z)_i = \theta(z_i)$. *If* θ *is decreasing, ceteris paribus, then* $g \circ \theta \in \mathcal{G}^{-\varepsilon}(\widetilde{\mathcal{W}})$.

For instance the problem $\mathcal{W}(x) = F(\max_{i\in[n]}\{x_i\})$ reduces to $\widetilde{\mathcal{W}}(z) = \max_{i\in[n]}\{z_i\}$ by the change $x_i = F^{-1}(z_i)$; and $\mathcal{W}(x) = \min_{i\in[n]}\{x_i\}$ reduces to $\widetilde{\mathcal{W}}(z) = \max_{i\in[n]}\{z_i\}$ by the change of variable $x_i = -z_i$.

3.3 Lipschitz and differentiability properties

They are key to the characterisation results in sections 6,7, 8.

Lemma 3.8 *Fix* $g \in \mathcal{G}^+$. *For any* x_i, x_i' *and any contact profile* (x_i, x_{-i}) *of* g *at* x_i *we have*

$$g(x_i') - g(x_i) \geq \mathcal{W}(x_i', x_{-i}) - \mathcal{W}(x_i, x_{-i}) \tag{8}$$

and the opposite inequality if $g \in \mathcal{G}^-$.

Proof In the inequality

$$g(x_i') + \sum_{j\neq i} g(x_j) \geq \mathcal{W}(x_i', x_{-i})$$

we replace each term $g(x_j)$ by $\mathcal{W}(x_i, x_{-i}) - g(x_i) - \sum_{k\neq i,j} g(x_k)$ and rearrange it as follows

$$(n-1)(\mathcal{W}(x_i, x_{-i}) - g(x_i)) - (n-2)\sum_{j\neq i} g(x_j) \geq \mathcal{W}(x_i', x_{-i}) - g(x_i')$$

$$\Longleftrightarrow \mathcal{W}(x_i, x_{-i}) - g(x_i) + (n-2)(\mathcal{W}(x) - \sum_{[n]} g(x_j)) \geq \mathcal{W}(x_i', x_{-i}) - g(x_i')$$

The term in parenthesis is zero by our choice of x_{-i} so we are done. ■

Our last general result is critical to both Theorems 6.1 and 7.1, where it is only used in a one-dimensional interval of types. But its statement and proof are just as easy when $\mathcal{X}$ is a multidimensional interval.

Lemma 3.9
i) Suppose K *is a positive constant,* $\mathcal{X} \subset \mathbb{R}^A$ *and the function* $\mathcal{W}$ *is* K*-Lipschitz in each* x_i, *uniformly in* $x_{-i} \in \mathcal{X}^{[n-1]}$. *Then so is each tight guarantee* $g \in \mathcal{G}^\varepsilon$ for $\varepsilon = +, -$.
ii) Suppose $\mathcal{X} = [\boldsymbol{L}, \boldsymbol{H}]$ *is the interval* $\boldsymbol{L} \leq x \leq \boldsymbol{H}$ *in* $\mathbb{R}^A$. *We fix* $x_i \in \mathcal{X}$, *a tight guarantee* $g \in \mathcal{G}^\varepsilon$ for $\varepsilon = +, -$ *and a contact profile* (x_i, x_{-i}) *of* g *at* x_i. *If for some* $a \in A$, g *and* $\mathcal{W}(\cdot, x_{-i})$ *are both differentiable in* x_{ia} *at* x_i, *we have*

if $\boldsymbol{L}_a < x_{ia} < \boldsymbol{H}_a$

$$\frac{dg}{dx_{ia}}(x_{ia}) = \frac{\partial \mathcal{W}}{\partial x_{ia}}(x_i, x_{-i}) \tag{9}$$

if $x_{ia} = \boldsymbol{L}_a$ and $g \in \mathcal{G}^-$, or $x_{ia} = \boldsymbol{H}_a$ and $g \in \mathcal{G}^+$

$$\frac{dg}{dx_{ia}}(x_{ia}) \leq \frac{\partial \mathcal{W}}{\partial x_{ia}}(x_i, x_{-i})$$

if $x_{ia} = \boldsymbol{H}_a$ and $g \in \mathcal{G}^-$, or $x_{ia} = \boldsymbol{L}_a$ and $g \in \mathcal{G}^+$

$$\frac{dg}{dx_{ia}}(x_{ia}) \geq \frac{\partial \mathcal{W}}{\partial x_{ia}}(x_i, x_{-i})$$

Proof *Statement i)* If $g \in \mathcal{G}^-$ inequality (8) and the Lipschitz assumption imply $g(x_i) - g(x'_i) \leq K\|x_i - x'_i\|$ (where $\|\cdot\|$ is the norm w. r. t. which $\mathcal{W}$ is Lipschitz). Exchanging the roles of x_i and x'_i gives $g(x'_i) - g(x_i) \leq K\|x'_i - x_i\|$ and the conclusion.

Statement ii) Note that if the functions f, g of one real variable z are differentiable at some z_0 in the interior of their common domain and the inequality $f(z) - f(z_0) \geq g(z) - g(z_0)$ holds for z close enough to z_0, then their derivatives at z_0 coincide. By inequality (8) we can apply this to the functions $x_{ia} \to g(x_i)$ and $x_{ia} \to \mathcal{W}(x_i, x_{-i})$, which proves (9). The last two inequalities are equally easy to deduce from (8). ■

For a fixed coordinate $a \in A$ the Lipschitz property in statement *i*), that we call *uniformly Lipschitz* by a slight abuse of terminology[8], implies that g is differentiable in x_{ia} almost everywhere in $[\boldsymbol{L}_a, \boldsymbol{H}_a]$. All our examples in sections 5,6,7 involve functions $\mathcal{W}$ uniformly Lipschitz in this sense, therefore all corresponding tight guarantees are differentiable almost everywhere in each coordinate of x_i.

Corollary to Lemma 3.9 *Suppose $\mathcal{X} = [L, H] \subset \mathbb{R}$, $\mathcal{W}$ is differentiable in $[L, H]^{[n]}$. Then for $\varepsilon = +, -$ the tight guarantees in $\mathcal{G}^\varepsilon$ are characterised by their contact set $\mathcal{C}(g)$: for any two different $g, h \in \mathcal{G}^\varepsilon$ we have $\mathcal{C}(g) \neq \mathcal{C}(h)$.*

Moreover any (true) convex combination of two or more guarantees in $\mathcal{G}^\varepsilon$ stays in $\boldsymbol{G}^\varepsilon$ but leaves $\mathcal{G}^\varepsilon$: $]g, h[\cap \mathcal{G}^\varepsilon = \varnothing$.

Proof. By statement *ii*) in the Lemma if $\mathcal{C}(g) = \mathcal{C}(h)$ we get $\frac{dg}{dx} = \frac{dh}{dx}$ in the interval $]L, H[$ so g and h differ by a constant, and if the constant is not zero one of g, h is not tight.

For the second statement suppose that $\mathcal{G}^-$ contains g, h and $\frac{1}{2}(g+h)$, all different. Fix $x_i \in]L, H[$ and a contact profile $(x_i, \widetilde{x}_{-i})$ of $\frac{1}{2}(g+h)$ at x_i. Clearly $\widetilde{x}_{-i}$ is also a contact profile of g and of h at x_i. Again by statement *ii*) this implies $\frac{dg}{dx_i}(x_i) = \frac{dh}{dx_i}(x_i) = \partial_i \mathcal{W}(x_i, \widetilde{x}_{-1})$ almost surely in $x_i \in]L, H[$. We conclude that $g - h$ is a constant and get a contradiction of $g \neq h$. The argument for larger convex combinations with general weights is entirely similar. ■

4 Modular functions $\mathcal{W}$

In this class of benefit and cost functions that includes most of our examples, the analysis of tight guarantees simplifies.

The type space $\mathcal{X}$ is a compact subset of $\mathbb{R}^A$ for Proposition 4.1, a compact interval in $\mathbb{R}^A$ for Proposition 4.2, and a one-dimensional interval in Proposition 4.3.

[8] Because we only require the Lipschitz property in each coordinate x_i.

Definition 4.1 *We call $\mathcal{W}$ supermodular* if for $i, j \in [n]$ and x, x' in $\mathcal{X}^{[n]}$ such that $x_k = x'_k$ for all $k \neq i, j$ we have

$$\{x_i \leq x'_i, x_j \leq x'_j\} \Longrightarrow \mathcal{W}(x'_i, x_j; x_{-i,j}) + \mathcal{W}(x_i, x'_j; x_{-i,j}) \leq \mathcal{W}(x) + \mathcal{W}(x') \tag{10}$$

We say that $\mathcal{W}$ is strictly supermodular if whenever $(x_i, x_j) \ll (x'_i, x'_j)$ the RH of (10) is strict. And $\mathcal{W}$ is *submodular or strictly so if the opposite inequalities holds under the same premises. A modular function is one that is either supermodular or submodular.*

An equivalent definition of supermodularity is useful too: for $i, j \in [n]$ and x, x' in $\mathcal{X}^{[n]}$ s. t. $x_j \leq x'_j$ for all j

$$\mathcal{W}(x'_i, x_{-i}) - \mathcal{W}(x_i, x_{-i}) \leq \mathcal{W}(x'_i, x'_{-i}) - \mathcal{W}(x_i, x'_{-i}) \tag{11}$$

The Appendix 9.3 lists other well known properties of the partial derivatives of modular functions that are useful in some of the long proofs.

4.1 the unanimity guarantee of modular functions

Proposition 4.1 *If $\mathcal{W}$ is supermodular the unanimity function (5) is the unique tight upper guarantee: $\mathcal{G}^+ = \{una\}$. It is the unique tight lower guarantee if $\mathcal{W}$ is submodular: $\mathcal{G}^- = \{una\}$.*

Notation: when the ordering of the coordinates does not matter $(z; \overset{k}{y})$ represents the $(k+1)$-vector where one coordinate is z and k coordinates are y.

Proof Fix $\mathcal{W}$ supermodular. For $n = 2$ the statement $una \in \mathcal{G}^+$ follows from$una \in \boldsymbol{G}^+$ and amounts to

$$\mathcal{W}(x_1, x_2) \leq \frac{1}{2}(\mathcal{W}(x_1, x_1) + \mathcal{W}(x_2, x_2))$$

a consequence of supermodularity and $\mathcal{W}(x_1, x_2) = \mathcal{W}(x_2, x_1)$.

By induction we assume the statement is true up to $(n-1)$ agents and fix a n-person supermodular function $\mathcal{W}$ and a profile $x \in \mathcal{X}^{[n]}$. As una is an upper guarantee of the $(n-1)$-benefit function $\mathcal{W}(\cdot; x_i)$ we have, for all i and x_i

$$\mathcal{W}(x) \leq \frac{1}{n-1} \sum_{j \in [n] \smallsetminus \{i\}} \mathcal{W}(x_i; \overset{n-1}{x_j}) \Longrightarrow n\mathcal{W}(x) \leq \frac{1}{n-1} \sum_{(i,j) \in P} \mathcal{W}(x_i; \overset{n-1}{x_j}) \tag{12}$$

where P is the set of ordered pairs (i, j) in $[n]$.

Fix now a pair i, j and apply the same property of una for $\mathcal{W}(\cdot; x_j)$ at the profile $(x_i, \overset{n-2}{x_j})$:

$$\mathcal{W}(x_i; \overset{n-1}{x_j}) \leq \frac{1}{n-1}((n-2)\mathcal{W}(\overset{n}{x_j}) + \mathcal{W}(x_j; \overset{n-1}{x_i}))$$

Summing up both sides over $(i, j) \in P$ and writing S for the summation in the RH inequality of (12) gives

$$S \leq (n-2) \sum_{j=1}^{n} \mathcal{W}(\overset{n}{x_j}) + \frac{1}{n-1} S \Longrightarrow S \leq (n-1) \sum_{j=1}^{n} \mathcal{W}(\overset{n}{x_j})$$

Combining the RH of (12) with the latter inequality concludes the proof.

The proof for a submodular $\mathcal{W}$ exchanges a few signs. ■

4.2 two canonical incremental guarantees

We look now at tight guarantees on the other side of the unanimity one.

Wether $\mathcal{W}$ is modular or not, for each unanimity profile $(\overset{n}{x_i})$ there is a non empty subset of tight lower guarantees g^- for which $(\overset{n}{x_i})$ is a contact profile: $g^-(x_i) = una(x_i)$ (Lemma 3.6). For a strictly modular function we expect this set to be infinite, as described in Theorem 7.1 for two person problems. But there are two important exceptions at the when the set of types is a multi-dimensional interval.

Proposition 4.2 *Suppose $\mathcal{X}$ is an interval $[\boldsymbol{L}, \boldsymbol{H}] \subseteq \mathbb{R}^A$ and $\mathcal{W}$ is supermodular. Then $\mathcal{W}$ has exactly one tight lower guarantee with the unanimous contact profile $(\overset{n}{\boldsymbol{L}})$ and one with contact profile $(\overset{n}{\boldsymbol{H}})$, called respectively the left-incremental g_{inc} and right-incremental g^{inc}:* for all $x_i \in [\boldsymbol{L}, \boldsymbol{H}]$

$$
\begin{aligned}
g_{inc}(x_i) &= \mathcal{W}(x_i; \overset{n-1}{\boldsymbol{L}}) - \frac{n-1}{n}\mathcal{W}(\overset{n}{\boldsymbol{L}}) \\
g^{inc}(x_i) &= \mathcal{W}(x_i; \overset{n-1}{\boldsymbol{H}}) - \frac{n-1}{n}\mathcal{W}(\overset{n}{\boldsymbol{H}})
\end{aligned}
\tag{13}
$$

If $\mathcal{W}$ is submodular replace lower guarantee by upper guarantee in the statement.

In Example 2.1 these two guarantees are the end-points g_L^+ and g_H^+ of $\mathcal{G}^+$. Here too, if $\mathcal{W}$ is supermodular and $\mathcal{W}(x)$ a surplus, g_{inc} favors the types x_i close to $\boldsymbol{L}$ who get a share close to their best case $una(x_i)$, and g^{inc} favors those close to $\boldsymbol{H}$. Isomorphic comments obtain if $\mathcal{W}(x)$ is a cost and/or $\mathcal{W}$ is submodular.

Proof Fix $\mathcal{W}$ supermodular and check first that g_{inc} is a feasible lower guarantee. If $n = 2$ this follows at once from (10). If $n = 3$ we must show the following inequality for any x:

$$\mathcal{W}(x_1, \boldsymbol{L}, \boldsymbol{L}) + \mathcal{W}(x_2, \boldsymbol{L}, \boldsymbol{L}) + \mathcal{W}(x_3, \boldsymbol{L}, \boldsymbol{L}) \leq \mathcal{W}(x_1, x_2, x_3) + 2\mathcal{W}(\boldsymbol{L}, \boldsymbol{L}, \boldsymbol{L})$$

We use the symmetry of W to apply successively (10) and (11):

$$\mathcal{W}(x_1, \boldsymbol{L}, \boldsymbol{L}) + \mathcal{W}(\boldsymbol{L}, x_2, \boldsymbol{L}) \leq \mathcal{W}(x_1, x_2, \boldsymbol{L}) + \mathcal{W}(\boldsymbol{L}, \boldsymbol{L}, \boldsymbol{L})$$

$$\mathcal{W}(\boldsymbol{L}, \boldsymbol{L}, x_3) - \mathcal{W}(\boldsymbol{L}, \boldsymbol{L}, \boldsymbol{L}) \leq \mathcal{W}(x_1, x_2, x_3) - \mathcal{W}(x_1, x_2, \boldsymbol{L})$$

and sum up these two inequalities.

The argument for any n is now clear: in the desired inequality

$$\sum_{[n]} \mathcal{W}(x_i; \overset{n-1}{\boldsymbol{L}}) \leq \mathcal{W}(x) + (n-1)\mathcal{W}(\overset{n}{\boldsymbol{L}})$$

we replace the the first two terms on the LH by $\mathcal{W}(x_1, x_2; \overset{n-2}{\boldsymbol{L}}) + \mathcal{W}(\overset{n}{\boldsymbol{L}})$: by (*10*) this increases weakly the LH so it is enough to check

$$\mathcal{W}(x_1, x_2; \overset{n-2}{\boldsymbol{L}}) + \sum_3^n \mathcal{W}(x_i; \overset{n-1}{\boldsymbol{L}}) \leq \mathcal{W}(x) + (n-2)\mathcal{W}(\overset{n}{\boldsymbol{L}})$$

Next by (11) we replace the two first terms on the LH by $\mathcal{W}(x_1, x_2, x_3; \overset{n-3}{\boldsymbol{L}}) + \mathcal{W}(\overset{n}{\boldsymbol{L}})$ and so on.

Next g_{inc} is tight by by (13) and Lemma 3.5, because $(x_i; \overset{n-1}{\boldsymbol{L}})$ is a contact profile of g_{inc} at any x_i. Finally if another lower guarantee g^- is s. t. $g^-(\boldsymbol{L}) = \frac{1}{n}(\overset{n}{\boldsymbol{L}})$ we have for all x_i

$$g^-(x_i) + (n-1)g^-(\boldsymbol{L}) \leq \mathcal{W}(x_i; \overset{n-1}{\boldsymbol{L}}) \Longrightarrow g^-(x_i) \leq g_{inc}(x_i)$$

so g^- is either equal to g_{inc} or not tight.

The proofs for g^{inc} and/or submodular $\mathcal{W}$ are identical up to switching the relevant signs. ■

Call the function $\mathcal{W}$ monotonic if it is weakly increasing or weakly decreasing; then by Lemma 3.4 every tight guarantee g of $\mathcal{W}$ is monotonic as well, and we call the difference $|g(\boldsymbol{H}) - g(\boldsymbol{L})|$ its *spread.* In Example 2.1 the spread of g_p^+ is $H - p$, largest for g_L^+ and smallest for g_H^+. This observation generalises.

Lemma 4.1 *If $\mathcal{W}$ is weakly increasing and super or submodular, the two incremental guarantees have the smallest and largest spread among all tight guarantees on the other side of the unanimity one.*

Proof We fix $\mathcal{W}$ supermodular, and a tight guarantee $g^- \in \mathcal{G}^-$; by Lemma 3.4 g^- is weakly increasing. Pick a contact profile x_{-i} of g^- at $\boldsymbol{L}$ then apply successively Lemma 3.8 at $\boldsymbol{H}$ and $\boldsymbol{L}$, and supermodularity:

$$g^-(\boldsymbol{H}) - g^-(\boldsymbol{L}) \geq \mathcal{W}(\boldsymbol{H}, x_{-i}) - \mathcal{W}(\boldsymbol{L}, x_{-i}) \geq$$

$$\geq \mathcal{W}(\boldsymbol{H}, \overset{n-1}{\boldsymbol{L}}) - \mathcal{W}(\boldsymbol{L}, \overset{n-1}{\boldsymbol{L}}) = g_{inc}(\boldsymbol{H}) - g_{inc}(\boldsymbol{L})$$

Taking now a contact profile y_{-i} of g^- at $\boldsymbol{H}$ we have similarly

$$g^-(\boldsymbol{L}) - g^-(\boldsymbol{H}) \geq \mathcal{W}(\boldsymbol{L}, y_{-i}) - \mathcal{W}(\boldsymbol{H}, y_{-i}) \geq$$

$$\geq \mathcal{W}(\boldsymbol{L}, \overset{n-1}{\boldsymbol{H}}) - \mathcal{W}(\boldsymbol{H}, \overset{n-1}{\boldsymbol{H}}) = g^{inc}(\boldsymbol{L}) - g^{inc}(\boldsymbol{H})$$

therefore the spread of g^- is at least that of g_{inc} and at most that of g^{inc}. The submodular case is similar. ■

Remark 4.1: If $\mathcal{W}$ is strictly super or submodular and $\mathcal{X}$ is a real interval, then a tight guarantee g on the other side of *una* can have at most one unanimous contact $(\overset{n}{x_1})$, i. e., there cannot be two types x_1, x_2 s. t. $g(x_i) = una(x_i)$ for $i = 1, 2$.[9] Many examples where g has no unanimous contact profile are in Proposition 5.2 and Theorem 6.1.

4.3 implementing the incremental guarantees: the serial rules

We adapt to our model these well known sharing rules, originally introduced for the commons problem with substitutable inputs ([21], [28]), the object of the next section.

Definition 4.3 *Suppose $\mathcal{X}$ is an interval $[L, H] \subseteq \mathbb{R}$. The increasing Serial sharing rule (SER↑) $\varphi^{ser\uparrow}$ is defined by the combination of two properties a) it is symmetric in its variables and b) the share of agent i with type x_i is independent of other agents' larger shares.*[10]

When the agents are labelled by increasing types as $x_1 \leq x_2 \leq \cdots \leq x_n$ agent i's share is:

$$\varphi_i^{ser\uparrow}(x) = \frac{\mathcal{W}(x_1, \cdots, x_{i-1}, \overset{n-i+1}{x_i})}{n-i+1} - \sum_{j=1}^{i-1} \frac{\mathcal{W}(x_1, \cdots, x_{j-1}, \overset{n-j+1}{x_j})}{(n-j+1)(n-j)} \qquad (14)$$

[9] If there is we compute $\mathcal{W}(\overset{n}{x_1}) + \mathcal{W}(\overset{n}{x_2}) = ng(x_1) + ng(x_2) = (g(x_2) + (n-1)g(x_1)) + (g(x_1) + (n-1)g(x_2))$ and the latter term is bounded by $\mathcal{W}(x_2; \overset{n-1}{x_1}) + \mathcal{W}(x_1; \overset{n-1}{x_2})$ so we contradict the assumptions on $\mathcal{W}$.

[10] The share $\varphi_i(x)$ does not change if agent j's type changes from x_j to x'_j both weakly larger than x_i.

We omit this computation for brevity: see the details in ([20]) where this is equation (6).

The decreasing Serial rule SER↓ is defined symmetrically by property a) and b)* agent i' share is independent of other agents' smaller shares. It is given by the same expression (14) if we label the agents by decreasing types.

Proposition 4.3 *Fix a supermodular function $\mathcal{W}$ in $[L,H]^{[n]}$.*
The SER↑ rule implements both the left-incremental and unanimity guarantees g_{inc}, una. The SER↓ rule implements both g^{inc} and una.

The isomorphic statement for submodular functions exchanges left- and right- incremental guarantees.

Proof in Appendix 9.4.

5 Substitute inputs

In this section the function $\mathcal{W}$ takes the form $\mathcal{W}(x) = F(x_N)$. The classic interpretation is a production function F in which the agents' non negative inputs x_i are perfect substitutes; alternatively x_i is a demand of output and $F(x_N)$ is the cost of meeting total demand. This problem is supermodular if (and only if) F is convex and submodular if F is concave. In the above interpretations F is increasing, but none of the results in this section require this assumption; in the facility location Example 5.2 below F is not monotonic.

5.1 stand alone guarantees

For a general function $\mathcal{W}$ a *stand alone* guarantee is one that takes the form $g(x_i) = \mathcal{W}(x_i, c) - \gamma$ where $c \in \mathcal{X}^{[n-1]}$ and $\gamma \in \mathbb{R}$ are constant. If $\mathcal{W}$ is modular the left and right incremental guarantees are prime examples of tight stand alone guarantees (Proposition 4.2); we will find many more in Section 6.1.

In the substitute inputs model we find $n-2$ additional tight stand alone guarantees linking the two incremental ones. Types x_i vary in the real interval $[L,H]$ and the domain of F is $[nL, nH]$.

Proposition 5.1
i) If F is convex in $[nL, nH]$ the supermodular commons $\mathcal{W}(x) = F(x_N)$ admits n tight lower guarantees $g_{\ell,h}$, where $\ell, h \in \mathbb{N} \cup \{0\}$ are s. t. $\ell + h = n-1$: for $x_i \in [L,H]$

$$g_{\ell,h}(x_i) = F(x_i + (\ell L + hH)) - \frac{1}{n}\{\ell F((\ell+1)L + hH) + hF(\ell L + (h+1)H)\} \tag{15}$$

In particular $g_{n-1,0} = g_{inc}$ and $g_{0,n-1} = g^{inc}$ ((13)).
ii) The gap $una(x_i) - g_{\ell,h}(x_i)$ is minimal at the benchmark type $x_i = \frac{1}{n-1}(\ell L + hH)$.
iii) If F is strictly convex only g_{inc} and g^{inc} have a unanimous contact point.

If F is concave (15) defines n tight upper guarantees with the same properties for the gap $g_{\ell,h} - una$ and contact points.

Proof in the Appendix 9.5.

Example 5.1: *Commons with complementary inputs*

A project wil return one unit of surplus if and only if all agents succeed in completing their own part. Agent i's effort $x_i \in [L,H]$ is also the probability that i is successful, and $[L,H] \subset]0,1[$. The agents share the expected return

$$\mathcal{W}(x) = x_1 x_2 \cdots x_n \text{ for } x \in [L,H]^{[n]}$$

The function $\mathcal{W}$ is supermodular so $una(x_i) = \frac{1}{n}x_i^n$ is the single tight upper bound on type x_i's share: a larger share is feasible only if x_i gets a free ride from some higher effort types.

By Lemma 3.7 the change of variable $x_i = e^{z_i}$ transforms $\mathcal{W}$ into $\widetilde{\mathcal{W}}(z) = e^{z_N}$ and the guarantees (15) for $\widetilde{\mathcal{W}}$ correspond for $\mathcal{W}$ to n tight lower guarantees linear in type:

$$g_{\ell,h}(x_i) = L^\ell H^h(x_i - \frac{1}{n}(\ell L + hH))$$

$$\begin{aligned} g_{inc}(x_i) &= L^{n-1}(x_i - \frac{n-1}{n}L) \\ g^{inc}(x_i) &= H^{n-1}(x - \frac{n-1}{n}H) \end{aligned}$$

We see that $g_{inc}(x_i) \geq \frac{1}{n}L^n$: providing even the minimal effort L guarantees the share $una(L) = \frac{1}{n}L^n$. But g^{inc} is much more generous to high effort, it guarantees $\frac{1}{n}H^n$ to the maximal effort H; this is feasible by charging cash penalties to all "slackers", defined as those with $x_i < \frac{n-1}{n}H$; for instance type L pays out $|g^{inc}(L)| = H^{n-1}(\frac{n-1}{n}H - L)$ in the worst case where all others provide maximal effort H.

The $n-2$ other guarantees $g_{\ell,h}$ allow the manager to adjust, along a grid increasingly fine as n grows, the critical effort level $\frac{1}{n}(\ell L + hH)$ guaranteeing a positive share of output.

5.2 tangent and hybrid guarantees

If the general function $\mathcal{W}$ is globally convex and differentiable in $[L,H]^{[n]}$ the tangent at any point $(\alpha, una(\alpha))$ of its unanimity graph defines a feasible but not necessarily tight lower guarantee $g_\alpha \in \mathbf{G}^-$: for $x_i \in [L,H]$

$$g_\alpha(x_i) = \frac{1}{n}\mathcal{W}(\overset{n}{\alpha}) + \partial_1\mathcal{W}(\overset{n}{\alpha})(x_i - \alpha)$$

Indeed the LS of (4) is now

$$\mathcal{W}(\overset{n}{\alpha}) + \partial_1\mathcal{W}(\overset{n}{\alpha})(x_N - n\alpha)) \leq \mathcal{W}(x)$$

precisely the tangent hyperplane inequality of $\mathcal{W}$ at $(\overset{n}{\alpha})$ because $\mathcal{W}$ is symmetric.

For the globally convex $\mathcal{W}(x) = F(x_N)$ we find that many of the tangents to the unanimity graph are *tight* lower guarantees: those touching that graph inside the subinterval of $[L,H]$ left after deleting $\frac{1}{n}$-th at each end. And on the deleted intervals we construct guarantees concatenating (parts of) a tangent and a stand alone guarantee. We obtain in this way a continuous line of tight guarantees with the two incremental ones at its endpoints.

Proposition 5.2: *If F is convex in $[nL, nH]$ the supermodular commons $\mathcal{W}(x) = F(x_N)$ admits the following tight lower guarantees g_α, where $\alpha \in [L,H]$ and $g_L = g_{inc}$, $g_H = g^{inc}$.*

i) If $\frac{n-1}{n}L + \frac{1}{n}H \leq \alpha \leq \frac{1}{n}L + \frac{n-1}{n}H$ the graph of g_α is tangent to that of una at $n\alpha$: for $L \leq x_i \leq H$

$$g_\alpha(x_i) = \frac{1}{n}F(n\alpha) + \frac{dF}{dx}(n\alpha)(x_i - \alpha) \quad (16)$$

ii) If $L \leq \alpha \leq \frac{n-1}{n}L + \frac{1}{n}H$ the graph starts as a tangent then takes a stand alone shape:
for $L \leq x_i \leq n\alpha - (n-1)L$

$$g_\alpha(x_i) = \frac{1}{n}F(n\alpha) + \frac{dF}{dx}(n\alpha)(x_i - \alpha)$$

for $n\alpha - (n-1)L \leq x_i \leq H$

$$g_\alpha(x_i) = F(x_i + (n-1)L) - \frac{(n-1)}{n}F(n\alpha) + (n-1)\frac{dF}{dx}(n\alpha)(\alpha - L) \tag{17}$$

iii) If $\frac{1}{n}L + \frac{n-1}{n}H \leq \alpha \leq H$ *the graph starts as a stand alone then turn into a tangent:*
for $L \leq x_i \leq n\alpha - (n-1)H$

$$g_\alpha(x_i) = F(x_i + (n-1)H) - \frac{n-1}{n}F(n\alpha) - (n-1)\frac{dF}{dx}(n\alpha)(H - \alpha)$$

for $n\alpha - (n-1)H \leq x_i \leq H$

$$g_\alpha(x_i) = \frac{1}{n}F(n\alpha) + \frac{dF}{dx}(n\alpha)(x_i - \alpha)$$

If F is concave in $[nL, nH]$ the same $g_\alpha, \alpha \in [L, H]$, are tight upper guarantees of $\mathcal{W}$.

Proof

Statement i) We already noted that g_α is in $\mathbf{G}^-$. For tightness we fix a type x_i and look for a vector x_{-i} such that $x_i + x_{N\setminus i} = n\alpha$: then (16) implies $\sum_{[n]} g_\alpha(x_j) = F(n\alpha)$ so (x_i, x_{-i}) is a contact profile of g_α at x_i by Lemma 3.5. Such x_{-i} exists if and only if $x_i + (n-1)L \leq n\alpha \leq x_i + (n-1)H$, precisely as we assume.

Statement ii) At a profile x where $x_i \leq n\alpha - (n-1)L$ for all i, we just saw that g meets the LH of (4). We check now this inequality for a profile x where the first t types are above $n\alpha - (n-1)L$, $t \geq 1$, and the other $n - t$ types (possibly zero) are below that bound.

In the LH of (4) a type x_i for $i \leq t$ affects the difference $F(x_i + x_{N\setminus i}) - F(x_i + (n-1)L)$; as $x_{N\setminus i} \geq (n-1)L$ the inequality in question is most demanding (the difference is smallest) if $x_i = n\alpha - (n-1)L$. Similarly a type x_j for $j > t$, if any, affects $\Delta = F(x_j + x_{N\setminus j}) - \frac{dF}{dx}(n\alpha)x_j$. The derivative of Δ w.r.t. x_j is weakly increasin; at $x_j = L$ it is $\frac{dF}{dx}(L + x_{N\setminus j}) - \frac{dF}{dx}(n\alpha)$, non negative because $t \geq 1$ implies $x_{N\setminus j} \geq n\alpha - (n-1)L + (n-2)L = n\alpha - L$. Therefore the inequality in question is most demanding if $x_j = L$. It is then enough to check

$$tg_\alpha(n\alpha - (n-1)L) + (n-t)g_\alpha(L) \leq F(tn\alpha - (t-1)nL)$$

$$\Longleftrightarrow \frac{dF}{dx}(n\alpha)(t-1)n(\alpha - L) \leq F(tn\alpha - (t-1)nL) - F(n\alpha)$$

which follows at once from the convexity of F.

Checking tightness. At a type $x_i \leq n\alpha - (n-1)L$ we have

$$x_i + (n-1)L \leq n\alpha \leq x_i + (n-1)(n\alpha - (n-1)L)$$

(replace x_i by L on the RH and rearrange). As in the proof of statement *i)* this implies the existence of a contact profile (x_i, x_{-i}) entirely inside $[L, n\alpha - (n-1)L]$. And at a type $x_i \geq n\alpha - (n-1)L$ we also see that $(x_i, \overset{n-1}{L})$ is a contact profile of g_α.

We omit the symmetric proof of statement *iii)*. ■

Example 5.2 *sharing the cost of the variance*

Agents choose a type x_i in $[0, 1]$ and must share (n times) the variance of their distribution:

$$\mathcal{W}(x) = \sum_{[n]} x_i^2 - \frac{1}{n}(\sum_{[n]} x_i)^2 \tag{18}$$

For instance x_i is i's location in $[0,1]$ and a public facility is located at the mean $\frac{1}{n}x_N$ of this distribution, to minimise the quadratic transportation costs to the facility: the total cost $\mathcal{W}(x)$ is precisely (18).

The problem is submodular and $una(x_i) \equiv 0$: no one should get a net profit but everyone can hope that his type is adopted by everyone else, in which case there is no cost. By statement $i)$ in Lemma 3.7 and a change of sign, every tight upper guarantee g^+ of $\mathcal{W}$ obtains from a tight lower guarantee g^* of $\mathcal{W}^*(x) = (x_N)^2$ as $g^+(x_i) = x_i^2 - \frac{1}{n}g^*(x_i)$.

The tangent lower guarantees of $\mathcal{W}^*$ (statement $i)$ in Proposition 5.2) are $g^*_\alpha(x_i) = n\alpha(2x_i - \alpha)$, corresponding to the tight upper guarantees $g^+_\alpha(x_i) = (x_i - \alpha)^2$ of $\mathcal{W}$ for $\alpha \in [\frac{1}{n}, \frac{n-1}{n}]$: the location α is "free", a type α never pays, and the worst cost share at other locations is exactly the travel cost to the benchmark. But if α is L or H the guarantees $g(x_i) = x_i^2$ and $(1-x_i)^2$ are not tight, and dominated by the incremental guarantees $g_{inc}(x_i) = \frac{n-1}{n}x_i^2$ and $g^{inc}(x_i) = \frac{n-1}{n}(1-x_i)^2$.

The tight guarantees g^+_h in Proposition 5.1, indexed by the single integer $h = 0, 1, \cdots, n-1$ are:

$$g^+_h(x_i) = \frac{n-1}{n}(x_i - \frac{h}{n-1})^2 + \delta_h$$

where $\delta_h = \frac{h(n-1-h)}{n^2(n-1)}$. As $\delta_h \leq \frac{1}{4n}$ if n is large and $\alpha \simeq \frac{h}{n-1}$ the guarantees g^+_α and g^+_h are similar: g^+_h is $\frac{n-1}{n}$flater than g^+_α and smaller at 0 and 1, but unlike g^+_α, it never vanishes.

5.3 a minimalist example

With this extremely simple two-piece linear convex cost function C we restrict attention to tight lower guarantee with the same two-piece linear shape and the kink at the same place as C: this simplification makes it easy to describe the constraints they entail. But we still find a two-dimensional continuum of such guarantees.

Three agents can each engage in a potentially polluting activity at a level x_i in $[0,2]$. Total activity x_{123} below 3 is costless, but requires cleaning at price 1 above 3: $C(x_{123}) = (x_{123} - 3)_+$.

The single tight upper guarantee $una(x_i) = (x_i - 1)_+$ means that a "clean" type $x_i \leq 1$ will never pay (but could be paid), because costs can only occur if the other two agents pollute more than $2x_i$; while "dirty" types $x_i \geq 1$ may pay the full cleaning cost of their excess pollution.

Proposition 5.1 proposes three stand alone lower guarantees: for all $x_i \in [0,2]$

$$g_{inc}(x_i) = 0 \ ; \ g^{inc}(x_i) = x_i - 1 \ ; \ g^*(x_i) = (x_i - 1)_+ - \frac{1}{3}$$

Under g_{inc} a clean type is also never paid, while a dirty type x_i may be lucky and pay nothing (if total pollution is at most 3) but will pay the full $x_i - 1$ if $x_j = x_k = 1$.[11]

The contrast is sharpest with g^{inc} charging for sure the full $x_i - 1$ to a dirty type, and allowing a clean type to be compensated as much as $|x_i - 1|$. The latter happens if $x_{jk} > 3$ (so x_j, x_k are both dirty): agents j, k together pay $x_{jk} - 2$ when the actual cost is only $x_{ijk} - 3$, therefore x_i receives the full savings $1 - x_i$ that she generated.

The compromise guarantee g^* (for $\ell = h = 1$ in the Proposition) caps the compensation for clean behavior to $\frac{1}{3}$. Fix any $x_i \leq 1$ and suppose $x_j = 0$ and $x_k = 2$: then there is no cleaning cost but x_k pays at least $\frac{2}{3}$, which is only possible if types x_i and 0 get $\frac{1}{3}$ each (and x_k pays exactly $\frac{2}{3}$).

The full set of two-piece linear lower guarantees with a kink at 1 is parametrised by $\alpha \in [0, \frac{1}{3}], \beta \in [3\alpha, 1]$ as follows

$$g^{\alpha,\beta}(x_i) = \begin{matrix} -(\alpha + (\beta - 3\alpha)|x_i - 1|) \\ \beta(x_i - 1) - \alpha \end{matrix} \quad \text{if} \quad \begin{matrix} x_i \leq 1 \\ x_i \geq 1 \end{matrix}$$

[11] Note that g_{inc} is characterised in $\mathcal{G}^-$ by the fact that the super-clean type 0 is never compensated.

where for $\alpha = 0$ and $\beta \in [0,1]$ we recognise the guarantees tangent to the unanimity function at 1 (Proposition 5.2) with end points $g^{0,0} = g_{inc}$ and $g^{0,1} = g^{inc}$. Then g^* strikes a plausible compromise where every type's cost share varies by exactly $\frac{1}{3}$ as a function of the other two types.

5.4 two familiar sharing rules and their guarantees

While Proposition 4.3 shows that the two Serial sharing rules implement the two incremental guarantees, we check that the guarantees implemented by the two prominent rules Average Returns and the Shapley value are not tight either from above or below.

We fix F strictly concave on $\mathbb{R}_+$ and such that $F(0) = 0$, and the set of types $[L, H]$ s. t. $0 \leq L < H$. The two sharing rules are:

Average Returns (AR): $\varphi_i^{ar}(x) = x_i AF(x_N)$, with the notation $AF(z) = \frac{F(z)}{z}$;[12]

Shapley value (Sha): $\varphi_i^{Sha}(x) = \mathbb{E}_S(F(x_i + x_S) - F(x_S))$, where the expectation is over $S, \varnothing \subseteq S \subseteq N \diagdown \{i\}$, uniformly distributed

Lemma 5.1 *For the Average and Shapley rules on* $[L, H]$:

$$g_{ar}^-(x_i), g_{Sha}^-(x_i) < una(x_i) = \frac{1}{n} F(nx_i)$$

for $x_i \in [L, H[$, *with equality at* H.

If $L > 0$ *we have*

$$g_{ar}^+(x_i), g_{Sha}^+(x_i) > g_{inc}(x_i) = F(x_i + (n-1)L) - \frac{n-1}{n} F(nL)$$

for $x_i \in]L, H]$ with equality at L.

If $L = 0$ *we have* $g_{ar}^+ = g_{Sha}^+ = g_{inc}$.

Proof For the AR rule the average return AF decreases strictly so that $g_{ar}^-(x_i) = x_i AF(x_i + (n-1)H) < x_i AF(nx_i)$ on $[0, H[$. Similarly on $[L, H]$ we have $g_{ar}^+(x_i) = F(x_i + (n-1)L) = g_{inc}(x_i) + \frac{n-1}{n} F(nL)$ so that g_{ar}^+ is only tight if $L = 0$ and in that case it is g_{inc}.

For the Shapley rule the strict concavity of F implies, for $x_i < H$

$$\begin{aligned} g_{Sha}^-(x_i) &= \frac{1}{n} \sum_{k=0}^{n-1} (F(x_i + kH) - F(kH)) \\ &< \frac{1}{n} \sum_{k=0}^{n-1} (F(x_i + kx_i) - F(kx_i)) = \frac{1}{n} F(nx) \end{aligned}$$

Similarly

$$g_{Sha}^+(x_i) = \frac{1}{n} \sum_{k=0}^{n-1} (F(x_i + kL) - F(kL))$$

If $L = 0$ this gives $g_{Sha}^+ = g_{inc}$. If $L > 0$ and $x_i > L$ we sum up, for $1 \leq k \leq n-1$, the inequalities

$$F(x_i + (k-1)L) - F(kL) > F(x_i + (n-1)L) - F(nL)$$

$$\Longrightarrow g_{Sha}^+(x_i) > \frac{n-1}{n}(F(x_i + (n-1)L) - F(nL)) + \frac{1}{n} F(x_i + (n-1)L) = g_{inc}(x_i)$$

[12] At the profile $\overset{n}{(0)}$ the definition needs adjusting, e. g. to equal split, but this does not affect the computations of worst and best cases.

6 Rank separable functions

In this section like the previous and next ones the domain of types $\mathcal{X}$ is an interval $[L, H] \subseteq \mathbb{R}$. The *decreasing order statistics* of the profile $x \in [L, H]^{[n]}$ is written $(x^k)_{k=1}^n$, so $x^1 = \max_i\{x_i\}$ and $x^n = \min_i\{x_i\}$. The statement "x_i is of rank k in profile x" is unambiguous if x_i is different from every other coordinate; otherwise we mean that x_i appears at rank k for some weakly increasing ordering of the coordinates of x.

Definition 6.1 *The function $\mathcal{W}$ on $[L, H]^{[n]}$ is called rank-separable if there exist n equicontinuous real valued functions w_k on $[L, H]$ s. t. $w_k(L) = w_\ell(L)$ for $k, \ell \in [n]$* and for $x \in [L, H]^{[n]}$

$$\mathcal{W}(x) = \sum_{k=1}^{n} w_k(x^k) \tag{19}$$

A rank-separable function is almost everywhere separably additive: this is true in the open cone of $[L, H]^{[n]}$ defined by the strict inequalities $x_1 < x_2 < \cdots < x_n$ and in the $n!$ isomorphic cones obtained by permuting the coordinates.

Recall that the equicontinuous functions w_k are differentiable almost everywhere (a. e.) in $[L, H]$.

Lemma 6.1 *The rank-separable function (19) is supermodular if and only if we have:* for $k \in [n-1]$ and a. e. in $x_i \in [L, H]$

$$\frac{dw_k}{dx}(x_i) \leq \frac{dw_{k+1}}{dx}(x_i) \tag{20}$$

It is submodular iff the opposite inequalities hold.

Proof in Appendix 9.6.

For instance $\max_{[n]}\{x_i\} = x^1$ (Example 2.1) is submodular while $\min_{[n]}\{x_i\} = x^n$ is supermodular.

To introduce our next result we write $\mathcal{W}(x) = x^1$ and write differently the tight upper guarantees g_p^+ in (2) (Proposition 2.1)

$$g_p^+(x_i) = (x_i - p)_+ + \frac{1}{n}p = \mathcal{W}(x_i, \overset{n-1}{p}) - \frac{n-1}{n}\mathcal{W}(\overset{n}{p})$$

At the beginning of section 5.1 we called this the *stand alone* form $g_{c,\gamma}(x_i) = \mathcal{W}(x_i, c) - \gamma$ where $c \in [L, H]^{[n-1]}$ is a $(n-1)$-profile of types and $\gamma \in \mathbb{R}$. Other examples are the incremental guarantees (Proposition 4.2) and the n guarantees in Proposition 5.1. In all cases (x_i, c) is a contact profile of *every* type x_i; applying this fact to each of the $n-1$ types c_k determines γ as a function of c and $\mathcal{W}$.

Definition 6.2 *Fix a function $\mathcal{W}$ and $c \in [L, H]^{[n-1]}$. If the function g_c* on $[L, H]$

$$g_c(x_i) = \mathcal{W}(x_i, c) - \frac{1}{n}(\sum_{k=1}^{n-1} \mathcal{W}(c_k, c)) \tag{21}$$

is a feasible (upper or lower) guarantee in $\mathbf{G}^\varepsilon$, we call it a general stand alone guarantee.

Then g_c is tight and (x_i, c) is a contact profile for all x_i.

Verifying the contact property $g_c(x_i) + \sum_1^{n-1} g_c(c_k) = \mathcal{W}(x_i, c)$ in (21) is straightforward.

Theorem 6.1 *Fix a rank-separable and supermodular function $\mathcal{W}$. The set of its tight lower guarantees is given by (21) for all possible choices of c: $\mathcal{G}^-(\mathcal{W}) = \{g_c; c \in [L, H]^{[n-1]}\}$. If instead $\mathcal{W}$ is submodular, this is the set of its tight upper guarantees.*

Proof in Appendix 9.7.

If the parameter $c = (\overset{n-1}{c_0})$ is unanimous the tight guarantee $g_c(x_i) = \mathcal{W}(x_i; \overset{n-1}{c_0}) - \frac{n-1}{n}\mathcal{W}(\overset{n}{c_0})$ "touches" the unanimity guarantee at c_0, $g_c(c_0) = una(c_0)$: c_0 is the benchmark type p in Example 2.1, L for g_{inc} and H for g^{inc}. But if c is not unanimous, we do not expect the graph of these two functions to intersect.

Example 6.1 *sharing baby sitting costs*

Three agents with one child each need help starting at time 0 and ending at different times x_i in $[0, H]$. The hourly rate increases more than linearly in the number of children to supervise: \$1 for one child, \$3 for two and \$6 for all three. Keeping in mind $x^1 \geq x^2 \geq x^3$ the total, supermodular, cost is

$$\mathcal{W}(x) = x^1 + 2x^2 + 3x^3$$

The uncontroversial unanimity upper guarantee $una(x_i) = 2x_i$ charges the per person rate for three children. A pair c^-, c^+ defining a tight lower guarantee cuts an interval $[c^-, c^+]$ of "normal" demands for which type x_i pays the marginal cost 2 minus a fixed rebate; for lower demands the marginal cost is *higher* at 3 but the rebate is higher; and vice versa for demands larger than normal. We write $\delta = \frac{1}{3}(c^+ - c^-)$ for the per person normal rebate:

$$g_c^-(x_i) = \begin{matrix} 3x_i - (\delta + c^-) & & 0 \leq x_i \leq c^- \\ 2x_i - \delta & \text{if} & c^- \leq x_i \leq c^+ \\ x_i + \frac{1}{3}(2c^+ + c^-) & & c^+ \leq x_i \leq H \end{matrix}$$

so g_c^- touches una only if $c^- = c^+$.

Note that a low demand type can end up paid by the others: this happens if $c^- < \frac{1}{7}c^+$ for $x_i < \frac{1}{2}\delta$ (in the lowest range of x_i), and if $c^- > \frac{1}{7}c^+$ for $x_i < \frac{1}{3}(\delta + c^-)$ (in the middle range.

Example 6.2 *sharing a connection cost*

After each agent i chooses a location x_i in the interval $[L, H]$ they must share the cost of connecting them (e. g. by building a road) which we assume linear in the largest distance between agents: for $x \in [L, H]^{[n]}$

$$\mathcal{W}(x) = x^1 - x^n \tag{22}$$

Should an agent be penalised (pay more than the average) for being far away at the periphery of the distribution of agents, and if so, by how much?

The cost function $\mathcal{W}$ is submodular and the tight lower guarantee is $una(x_i) \equiv 0$: everyone's best case is to pay nothing (as in Example 5.2, the cost is zero at a unanimous profile). By Theorem 6.1 a tight upper guarantee involves the choice of $n - 1$ variables c_k.

For $n = 2$ equation (21) describes $\mathcal{G}^+$ as: $g_c^+(x_i) = |x_i - c|$ where $c \in [L, H]$. For $n \geq 3$ we see in (21) that only the largest and smallest values c^+ and c^- matter:

$$g_c(x_i) = (\max\{x_i, c^+\} - \min\{x_i, c^-\}) - \frac{n-1}{n}(c^+ - c^-)$$

Setting $\mu = \frac{1}{n}(c^+ - c^-)$ we develop this equation as follows: $g_c(x_i) = \mu$ if $c^- \leq x_i \leq c^+$; $g_c(x_i) = \mu + (c^- - x)$ if $L \leq x_i \leq c^-$; $g_c(x_i) = \mu + (x - c^+)$ if $c^+ \leq x_i \leq H$.

All types in the benchmark interval $[c^-, c^+]$ have the same worst cost share μ; a type outside this interval could pay, in addition to μ, the full connecting cost to the benchmark.

If $c^- = c^+ = c^*$ an agent locating at c^* pays nothing (irrespective of other agents' location) and $g_c(x_i) = |x_i - c^*|$. While if $(c^-, c^+) = (L, H)$ the worst cost share is $\frac{1}{n}(H - L)$ for everybody, compatible with the Equal-Split sharing rule, and many others.

Example 6.2A *facility location with linear transportation costs*

The optimal location is the median of the profile of types (as opposed to its mean in Example 5.2). Assume $n = 2q + 1$ is odd so the median is x^{q+1} and the agents share the cost

$$\mathcal{W}(x) = \sum_{k=1}^{q} x^k - \sum_{\ell=q+2}^{2q+1} x^\ell$$

a submodular and rank separable function similar to (22). Again the unanimity cost is zero and, surprisingly, the set $\mathcal{G}^+$ is two-dimensional and very similar to the one we just described: in the expression of g_c simply replace c^+ by c^q and $c^-$by c^{q+1}, and the coefficient $\frac{n-1}{n}$ by $\frac{q+1}{q}$ (we omit the tedious computations).

In our last example the function $\mathcal{W}_k$ is *neither sub nor supermodular.*

Example 6.3 *production with quota*

Fix n and a quota $k, 2 \leq k \leq n-1$. Agent i inputs the effort x_i: to achieve the output $y = F(z)$ we need at

least k agents contributing an effort at least z: for $x \in [L, H]^{n]}$

$$\mathcal{W}_k(x) = F(x^k) \text{ for } x \in [L, H]^{n]} \tag{23}$$

If $k = 1$ this is the submodular Example 2.1, up to a change of variable, and if $k = n$ this is its supermodular mirror image. For other values of k $\mathcal{W}_k$ is not modular.

Clearly $una(x_i) = \frac{1}{n}F(x_i)$ is neither a lower guarantee nor an upper guarantee: there is now a one dimensional choice of tight guarantees on both sides of (4). The set $\mathcal{G}_k^+$ is parametrised by $p \in [L, H]$:

$$g_{k,p}^+(x_i) = \frac{1}{n}F(p) + \frac{1}{k}(F(x_i) - F(p))_+$$

and $\mathcal{G}_k^-$ is similarly parametrised by $q \in [L, H]$:

$$g_{k,q}^-(x_i) = \frac{1}{n}F(q) + \frac{1}{n-k+1}(F(x_i) - F(q))$$

The proof, in Appendix 9.8, mimicks that of Proposition 2.1.

If $p = q = z^*$ this "standard" level of effort guarantees the share $\frac{1}{n}F(z^*)$. If the actual input x^k is below z^* the "slackers" inputting a sub-standard effort get on average less than $\frac{1}{n}F(z^*)$ if there are some hard working agents who get at least $> \frac{1}{n}F(x^k)$. Symetrically if x^k is above z^* the "slackers" cannot get more than the standard share $\frac{1}{n}F(z^*)$, and may get less if more than k agents input x_i larger than z^*.

7 Two person modular problems

In two person strictly modular problems with one-dimensional types we describe the tight solutions of system (4) on the other side of the unanimity by their contact set. For a tight guarantee g this set has the simple shape of a decreasing and occasionally multivalued function φ described in the next two Lemmas. Conversely we can pick any such function φ and integrate the differential equation $\frac{dg}{dx_i}(x_i) = \frac{\partial \mathcal{W}}{\partial x_i}(x_i, \varphi(x_i))$ (Lemma 3.9) to get an integral representation of the tight guarantee of which φ describe the contact set. This is the closed form solution of the functional inequalities (4).

Given a modular function $\mathcal{W}$ on $[L, H]^2$ and $g \in \mathcal{G}^\pm(\mathcal{W})$, a tight guarantee on either side of (4), we define the contact correspondence φ:

$$\varphi(x_1) = \{x_2 \in [L, H] | g(x_1) + g(x_2) = \mathcal{W}(x_1, x_2)\} \tag{24}$$

(non empty by Lemma 3.5) and write its graph $\Gamma(\varphi)$.

Lemma 7.1 *If $\mathcal{W}$ is supermodular, $g \in \mathcal{G}^-$* and $\Gamma(\varphi)$ *contains (x_1, x_2) and (x_1', x_2') s.t. $(x_1, x_2) \ll (x_1', x_2')$, then $(x_1, x_2'), (x_1', x_2) \in \Gamma(\varphi)$ as well, and $\mathcal{W}$ is not strictly supermodular.*

For a submodular function $\mathcal{W}$ replace $\mathcal{G}^-$ by $\mathcal{G}^+$.

Proof We sum up the two equalities in (24) for (x_1, x_2) and $(x_1\prime, x_2\prime)$:

$$\mathcal{W}(x_1, x_2) + \mathcal{W}(x_1', x_2') = \{g(x_1) + g(x_2')\} + \{g(x_1') + g(x_2)\} \leq \mathcal{W}(x_1, x_2') + \mathcal{W}(x_1', x_2)$$

Combined with the supermodular inequality (10) this gives an equality and the conclusion by Definition 4.1. As explained in Appendix 9.3 we conclude that $\mathcal{W}$ is locally additive. ■

Lemma 7.2 *Fix a strictly supermodular function $\mathcal{W}$ and a tight guarantee $g \in \mathcal{G}^-$ – or a submodular $\mathcal{W}$ and $g \in \mathcal{G}^+$ – with contact correspondence φ.*
i) $\Gamma(\varphi)$ is symmetric: $x_2 \in \varphi(x_1) \iff x_1 \in \varphi(x_2)$ for all x_1, x_2.
ii) φ is convex valued: $\varphi(x_1) = [\varphi^-(x_1), \varphi^+(x_1)]$, single-valued a.e., and upper-hemi-continuous (its graph is closed).
iii) φ^- and φ^+ are weakly decreasing and $x_1 \leq x_1' \Longrightarrow \varphi^-(x_1) \geq \varphi^+(x_1')$; φ is the u.h.c. closure of both φ^- and φ^+.
iv) $\varphi(L)$ contains H and $\varphi(H)$ contains L.
v) φ has a unique fixed point a: $a \in \varphi(a)$, and a is an end-point of $\varphi(a)$.

Proof in Appendix 9.9.

Theorem 7.1 *Fix a strictly super (resp. sub) modular function $\mathcal{W}$, continuously differentiable in $[L, H]^2$.*
i) For any correspondence φ as in Lemma 7.2, the following equation

$$g(x_1) = \int_a^{x_1} \partial_1 \mathcal{W}(t, \varphi(t)) dt + una(a) \tag{25}$$

defines a tight lower guarantee $g \in \mathcal{G}^-$ (resp. $\mathcal{G}^+$).
ii) Conversely if g is a guarantee in $\mathcal{G}^-$ (resp. $\mathcal{G}^+$) with contact correspondence φ (as in Lemma 7.2) then g takes the form (25).

Proof in Appendix 9.10.

So the sets $\mathcal{G}^\pm$ on the other side of unanimity are parametrised by a large set of functions φ. After choosing the benchmark type a which guarantees the share $una(a)$ we can pick any decreasing single-valued function $\overline{\varphi}$ from $[L, a]$ into $[a, H]$ mapping L to H, then fill the (countably many) jumps down to create the correspondence φ of which the graph connects (L, H) to (a, a), and finally extend φ to $[a, H]$ by symmetry of its graph around the diagonal of $[L, H]^2$.

We illustrate this embarrassement of riches in the commons with substitute inputs (Section 5).

Example 7.1 *Commons with substitutable inputs*

We have $\mathcal{W}(x) = F(x_1 + x_2)$ and F is strictly concave on $[0, 1]$.

The contact correspondence of the incremental guarantees g_{inc} is $\varphi_{inc}(0) = [0, 1]; \varphi_{inc}(x_1) = 0$ for $x_1 \in]0, 1]$; and φ^{inc} simply exchange the role of 0 and 1.

Proposition 5.1 has no bite for $n = 2$. Statement $i)$ in Proposition 5.2 delivers a single full tangent guarantee $g_{\frac{1}{2}}$ in (16) with the anti-diagonal contact function $\varphi_{\frac{1}{2}}(x_i) = 1 - x_i$. The contact functions of the guarantees in statements $ii)$ and $iii)$ are two-piece linear. For instance if $\alpha \in [0, \frac{1}{2}]$: $\varphi_\alpha(0) = [2\alpha, 1]$; $\varphi_\alpha(x_i) = 2\alpha - x_i$ on $]0, 2\alpha]$ and $\varphi_\alpha(x_i) = 0$ on $[2\alpha, 1]$.

To find new tight upper guarantees connecting g_{inc} and g^{inc} we pick φ with a similar piecewise constant graph. For $\beta \in [0, 1]$ define $\varphi_\beta \equiv 1$ on $[0, \beta[$; $\varphi_\beta(\beta) = [\beta, 1]$; $\varphi_\beta(x_i) = \beta$ on $]\beta, 1[$; $\varphi_\beta(1) = [0, \beta]$. Equation (25) gives

$$g_\beta(x_i) = \begin{array}{c} F(x_i+1) - F(\beta+1) + \frac{1}{2}F(2\beta) \\ F(x_i+\beta) - \frac{1}{2}F(2\beta) \end{array} \text{ if } \begin{array}{c} x_i \le \beta \\ x_i \ge \beta \end{array}$$

concatenating two different stand alone-like pieces, connected at $x_i = \beta$ where they touch the unanimity graph but, unlike the g_α-s in Proposition 5.2, the connection is not smooth.

Taking the symmetric of φ_β around the anti-diagonal we find, after similar computations, a second family of non smooth concatenations of stand alone-like pieces:

$$g_\beta(x_i) = \begin{array}{c} F(x_i+\gamma) - \frac{1}{2}F(2\gamma) \\ F(x_i) - F(\gamma) + \frac{1}{2}F(2\gamma) \end{array} \text{ if } \begin{array}{c} x_i \le \gamma \\ x_i \ge \gamma \end{array}$$

8 Concluding comments

We start with two open questions.

extending Theorem 7.1 for $n \ge 3$ The key for two agent problems is the deep understanding of the contact correspondence of any tight guarantee (Lemmas 7.1, 7.2). We could not gain a similar understanding of this correspondence with three or more agents. In particular Lemma 7.2 shows that in a *two agent* problem the contact set of *every* tight guarantee g in $\mathcal{G}^\varepsilon$ intersects the diagonal (g touches *una*): this gives the starting point of the integral equation (25). But we saw in Proposition 5.1 and Theorem 6.1 when $n \ge 3$ many tight guarantees of which the contact set does not intersect the diagonal.

multi-dimensional types The general results in Section 3 apply to functions $\mathcal{W}$ of m real variables x_i for any m, and so do the Propositions 4.1 and 4.2 for general modular functions. On the way to further develop the multidimensional analysis we run into an extremely challenging *decentralisation* question.

The following claim is obvious from the definitions and Lemma 3.5. Suppose each type has two components $x_i = (x_i^1, x_i^2) \in \mathcal{X}^1 \times \mathcal{X}^2 = \mathcal{X}$ and pick two functions $\mathcal{W}_1$ on $\mathcal{X}^{1[n]}$ and $\mathcal{W}_2$ on $\mathcal{X}^{2[n]}$. If $g_i^\varepsilon \in \mathcal{G}^\varepsilon(\mathcal{W}_i)$ for some $\varepsilon = +, -$ and both $i = 1, 2$, then $g_1^\varepsilon + g_2^\varepsilon$ is a tight guarantee of the function $\mathcal{W}$ adding the two independent problems as $\mathcal{W}(x) = \mathcal{W}_1(x^1) + \mathcal{W}_2(x^2)$ for $x \in \mathcal{X}$.

We do not know for which domain of functions $\mathcal{W}$ the converse decentralisation property holds: *every tight guarantee g^ε of $\mathcal{W}_1+\mathcal{W}_2$ (two functions in the domain) is the sum of two tight guarantees in the component problems.*

The answer eludes us even for the specific problem of assigning more than one indivisible object and cash transfers when utilities are additive over objects (and linear in money): the corresponding function $\mathcal{W}$ is the sum of problems $\mathcal{W}_a(x^a) = \max_{i\in[n]}\{x_i^a\}$ over several objects a. With much sweat we showed that the decentralisation property holds for two agents and two objects![13]

relation to optimal transport The tight guarantees g^- and g^+ to a given symmetric function $\mathcal{W}$ are its best approximations by symmetric additively separable functions from above and below. There is a clear formal connection[14] to the celebrated Optimal Transport problem ([35], [11]), specifically to its dual formulation as the Kantorovitch- Rubinstein Lemma:

$$\max_{\Pi:\Pi_i=\lambda_i}\{\int \mathcal{W}(x)d\Pi(x)\} = \min_{g_i:\sum_i g_i(x_i)\ge\mathcal{W}(x)}\{\sum_i \int g_i(x_i)d\lambda_i\}$$

[13] The proof is available upon request from the authors.
[14] We thank Fedor Sandomirskiy for pointing it out.

where $\mathcal{W}(x)$ is the abstract transport cost, and Π the transportation protocol with fixed marginals λ_i over the n coordinates of x.

The symmetry assumption is central to our approach: it restricts the marginals λ_i to be identical and the function Π symmetric, which is not the case in a standard Monge transportation problem or the matching models discussed in ([11]). We believe that the insights of that literature for fair division problems can be very helpful.

a research program Evidently the concept of tight guarantees applies to many more common property resources problems than those captured by our transferable utility model with modular benefits or costs and mostly one dimensional types.

More general abstract descriptions of the resources map profiles of types to subsets of feasible utility (or disutility) profiles. Horizontal equity confirms the prominent role of the unanimity guarantee and the complexity of the menu of tight guarantees will increase.

The selection of a subset of pairs of tight guarantees will require additional context-free tools. Even in our simple model we can choose $g^{\pm}$ so that $(una, g^{\pm})$ minimises the largest gap over all types, or, as in the otpimal transport problem, the expected gap w.r.t. some given distribution of types.

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9 Appendix: missing proofs

9.1 Lemma 3.5

Step 1: upper-hemi-continuity We fix $g \in \mathcal{G}^-$ and check that it is u.h.c.. If it is not, there is in $\mathcal{X}$ some x_1, a sequence $\{x_1^t\}$ converging to x_1, and some $\delta > 0$ such that $g(x_1^t) \geq g(x_1) + \delta$ for all t. Then we have, for any $x_{-1} \in \mathcal{X}^{[n-1]}$

$$\mathcal{W}(x_1^t, x_{-1}) \geq g(x_1^t) + \sum_{i=2}^{n} g(x_i) \geq (g(x_1) + \delta) + \sum_{i=2}^{n} g(x_i)$$

Taking the limit in t of $\mathcal{W}(x_1^t, x_{-1})$ and ignoring the middle term we see that we can increase g at x_1 without violating (4), a contradiction of our assumption $g \in \mathcal{G}^-$.

Step 2: statement ii) "If" is clear. For "only if" we fix $g \in \mathcal{G}^-$ and show that it meets property (*7*). For any $x_1 \in \mathcal{X}$ define

$$\delta(x_1) = \min_{x_{-1} \in \mathcal{X}^{[n-1]}} \{\mathcal{W}(x_1, x_{-1}) - \sum_{[n]} g^-(x_i)\}$$

and note that this minimum is achieved at some $\overline{x}_{-1}$ because the function $x_{-1} \to \sum_{i=2}^{n} g^-(x_i)$ is u.h.c. (step 1). Moreover $\delta(x_1)$ is non negative.

If $\delta(x_1) = 0$ property (7) holds at $\overline{x}_{-1}$. If $\delta(x_1) > 0$ we can increase g at x_1 to $g(x_1) + \delta(x_1)$, everything else equal, to get a guarantee dominating g.

Step 3: lower-hemi-continuity We fix $g \in \mathcal{G}^-$ and check that it is l.h.c.. By assumption $\mathcal{W}$ is equi-continuous in its first variable, uniformly in the others:

$$\forall \eta > 0, \exists \theta > 0, \forall x_1, x_1^*, x_{-1} : ||x_1 - x_1^*|| \leq \theta \Rightarrow \mathcal{W}(x_1, x_{-1}) \leq \mathcal{W}(x_1^*, x_{-1}) + \eta \qquad (26)$$

If g is not l.h.c. there is some x_1 and $\{x_1^t\}$ converging to x_1 and $\delta > 0$ s.t. $g(x_1^t) \leq g(x_1) - \delta$ for all t. Pick θ for which (26) holds with $\eta = \frac{1}{2}\delta$ and t large enough that $||x_1^t - x_1|| \leq \theta$: then for any x_{-1} we have

$$g(x_1) + \sum_{i=2}^{n} g(x_i) \leq \mathcal{W}(x_1, x_{-1}) \leq \mathcal{W}(x_1^t, x_{-1}) + \frac{1}{2}\delta$$

Replacing $g(x_1)$ with $g(x_1^t) + \delta$ gives $g(x_1^t) + \sum_{i=2}^{n} g(x_i) \leq \mathcal{W}(x_1^t, x_{-1}) - \frac{1}{2}\delta$ for any x_{-1}: this contradicts the contact property (7) for x_1^t.

9.2 Lemma 3.6

Proof *Statement i)* Fix $\varepsilon = -$, an arbitrary $\widetilde{x}_1 \in \mathcal{X}$ and write $B(\widetilde{x}_1, r)$ for the closed ball of center $\widetilde{x}_1$ and radius r. Use the notation $\Delta(x) = \sum_1^n una(x_i) - \mathcal{W}(x)$ to define the function

$$\delta(x_1) = \max\{\Delta(x_1, x_{-1}) : \forall i \geq 2, x_i \in B(\widetilde{x}_1, d(x_1, \widetilde{x}_1))\}$$

It is clearly continuous, non negative because $\Delta(x_1, x_{-1}) = 0$ if $x_i = x_1$ for $i \geq 2$, and $\delta(\widetilde{x}_1) = 0$. Define $g = una - \delta$ and check that g is the desired lower guarantee of $\mathcal{W}$. At an arbitrary profile $x = (x_i)_1^n$ choose x_{i^*} s.t. $d(\widetilde{x}_1, x_{i^*})$ is the largest: this implies $\delta(x_{i^*}) \geq \Delta(x)$. Combining this with $\delta(x_i) \geq 0$ for $i \neq i^*$ gives $\sum_1^n \delta(x_i) \geq \Delta(x)$ which, in turn, is the LH inequality in (4) for g. As g is in $\boldsymbol{G}^-$, it is dominated by some $\widetilde{g}$ in $\mathcal{G}^-$(Lemma 3.1) and $\widetilde{g}(x_1) = una(\widetilde{x}_1)$ by inequality (6).

Statement ii) We assume that $\boldsymbol{G}^-$ does not contain una and check that $\boldsymbol{G}^-$ is not a singleton. This assumption and the continuity of $\mathcal{W}$ imply that for an open set of profiles $x \in \mathcal{X}^{[n]}$ we have $\sum_{[n]} una(x_i) > \mathcal{W}(x)$. Fix such an x and (by statement i)) pick for each i a tight guarantee g_i equal to una at x_i: these n guarantees are not identical.

9.3 Some properties of modular functions

Submodularity is preserved by positive linear combinations, but not by the maximum or minimum operation. For instance if n is odd, the median of profile x (Example 6.3) is the minimum of several submodular functions but is neither sub- nor super-modular.

Whenever the partial derivative $\partial_i \mathcal{W}(x)$ is defined in a neighborhood of x, supermodularity implies that it is weakly increasing in x_j for $j \neq i$. And if $\partial_i \mathcal{W}(x)$ is strictly increasing in x_j then $\mathcal{W}$ is strictly supermodular. The isomorphic statements for submodularity replaces increasing by decreasing.

Whenever $\partial_i \mathcal{W}(x)$ is differentiable almost everywhere, the supermodularity property can be written: for $i, j \in [n], i \neq j$, $\partial_{ij} \mathcal{W}(x) \geq 0$ a. e. in $x \in [L, H]^{[n]}$. For submodularity reverse the inequality.

A well known consequence of modularity is this: if $(x_i, x_j) \ll (x'_i, x'_j)$ and the RH of (*10*) is an equality, then in the interval $[(x_i, x_j), (x'_i, x'_j)]$ the function $(z_i, z_j) \to \mathcal{W}(z_i, z_j; x_{-i,j})$ is separably additive, and its cross derivative $\partial_{ij} \mathcal{W}(\cdot, \cdot; x_{-i,j})$ is identically zero. We say that $\mathcal{W}$ is *locally* i, j-*additive* at the profile x if there is a rectangular neighborhood of (x_i, x_j) in which $\partial_{ij} \mathcal{W}(\cdot; x_{-i,j})$ is zero.

A strictly modular function like $\mathcal{W}(x) = F(\sum_{[n]} x_i)$ in section 5, with F strictly convex or strictly concave, is not locally i, j-additive anywhere. But the submodular function $\mathcal{W}(x) = \max_i\{x_i\}$ (Example 2.1) is locally i, j-additive whenever $x_i \neq x_j$, hence almost everywhere, although it is clearly not globally i, j-additive!.

9.4 Proposition 4.3

We prove the statement for the serial↑ rule (14). By Lemma 3.3 it is enough to check the inequality $g_{inc}(x_i) \leq \varphi_i^{ser\uparrow}(x) \leq una(x_i)$ for all x.

Step 1. We show that $\varphi_i^{ser\uparrow}(x)$ increases (weakly) in all variables x_j such that $x_j \leq x_i$, i. e., for $j \leq i - 1$. This generalises Lemma 1 in [20].

If $\mathcal{W}$ is differentiable in $[L, H]^n$ we check this by computing the derivative $\partial_k \varphi_i^{ser\uparrow}$ for $k \leq i - 1$ in the LH of equation (14) and using the symmetry of $\mathcal{W}$:

$$\partial_k \varphi_i^{ser\uparrow}(x) = \frac{\partial_k \mathcal{W}(x_1, \cdots, x_{i-1}, \overset{n-i+1}{x_i})}{n-i+1} - \frac{\partial_k \mathcal{W}(x_1, \cdots, x_{k-1}, \overset{n-k+1}{x_k})}{n-k} - \sum_{j=k+1}^{i-1} \frac{\partial_k \mathcal{W}(x_1, \cdots, x_{j-1}, \overset{n-j+1}{x_i})}{(n-j+1)(n-j)}$$

Recall that the coordinates of x are weakly increasing. Because $\partial_k \mathcal{W}$ increases weakly in $x_j, j \neq k$, the numerator of each negative fraction is not larger than that of the first fraction. The identity $\frac{1}{n-i+1} = \frac{1}{n-k} + \sum_{j=k+1}^{i-1} \frac{1}{(n-j+1)(n-j)}$ concludes the proof.

Without the differentiability assumption the only step that requires an additional argument is the following consequence of supermodularity: as the coordinates of x increase weakly the term $\mathcal{W}(x) - \frac{1}{n-k+1}\mathcal{W}(x_1, \cdots, x_{k-1}, \overset{n-k+1}{x_k})$ increases weakly in x_k for each $k \leq n-1$. We omit the straightforward proof.

Step 2. By construction of $\varphi^{ser\uparrow}$ we have $\varphi_i^{ser\uparrow}(x) = \varphi_i^{ser\uparrow}(x_1, \cdots, x_{i-1}, \overset{n-i+1}{x_i})$ and by Step 1 it is enough to check that $g_{inc}(x_i)$ lower bounds $\varphi_i^{ser\uparrow}(x)$ at the profile $(\overset{i-1}{L}, \overset{n-i+1}{x_i})$ while *una* upper bounds it at $(\overset{n}{x_i})$. The latter follows from $\varphi_i^{ser\uparrow}(\overset{n}{x_i}) = una(x_i)$. Applying (14) we see that the desired lower bound reduces to

$$\mathcal{W}(\overset{n-1}{L}, x_i) \leq \frac{1}{n-i+1}\mathcal{W}(\overset{i-1}{L}, \overset{n-i+1}{x_i}) + \frac{n-i}{n-i+1}\mathcal{W}(\overset{n}{L})$$

$$\iff (n-i)(\mathcal{W}(\overset{n-1}{L}, x_i) - \mathcal{W}(\overset{n}{L})) \leq \mathcal{W}(\overset{i-1}{L}, \overset{n-i+1}{x_i}) - \mathcal{W}(\overset{n-1}{L}, x_i)$$

Finally we apply (11) to successively lower bound $\mathcal{W}(\overset{k}{L}, \overset{n-k}{x_i}) - \mathcal{W}(\overset{k+1}{L}, \overset{n-k-1}{x_i})$ by $\mathcal{W}(\overset{n-1}{L}, x_i) - \mathcal{W}(\overset{n}{L})$ for $k = (n-2), \cdots, (i-1)$ and sum up these inequalities.

9.5 Proposition 5.1

Step 1 The function $g_{\ell,h}$ defined by (15) is a lower guarantee: $g_{\ell,h} \in \mathbf{G}^-$.

We set $Z = \ell L + hH$ for easier reading. The feasibility inequality (4) applied to $g_{\ell,h}$ reads: for $x \in [L,H]^{[n]}$

$$\sum_{[n]} F(x_i + Z) \leq F(x_N) + \ell F(Z + L) + hF(Z + H) \tag{27}$$

We proceed by induction on n. There is nothing to prove if $n = 2$. For $n = 3$ we already know that $g_{2,0}$ and $g_{0,2}$ are in $\mathcal{G}^-$; for $g_{1,1}$ the inequality (27) is

$$\sum_{[3]} F(x_i + L + H) \leq F(x_{123}) + F(2L + H) + F(L + 2H) \tag{28}$$

Suppose $x_{12} \geq L + H$: then the convexity of F implies

$$F(x_3 + L + H) - F(2L + H) \leq F(x_{123}) - F(x_{12} + L)$$

Replacing $F(x_3 + L + H)$ in (28) by this upper bound and rearranging gives a more demanding inequality

$$F(x_1 + L + H) + F(x_2 + L + H) \leq F(x_{12} + L) + F(L + 2H)$$

following again from the convexity of F.

So we are done if $x_{ij} \geq L + H$ for any pair i, j. Suppose next $x_{ij} \leq L + H$ for all three pairs. Then we have for $i = 1, 2, 3$

$$x_{123}, 2L + H \leq x_i + L + H \leq L + 2H$$

and the uniform distribution on the triple $x_{123}, 2L + H, L + 2H$ is a mean-preserving spread of that on $(x_i + L + H)_{i \in [3]}$, which proves (28).

For the inductive argument we fix $n \geq 4$ and $g_{\ell,h}$ s. t. $\ell + h = n - 1$ and $\ell \geq 1$. We assume that (27) holds for $n-1$ agent problems and prove it for (ℓ, h).

Suppose $x_{N \setminus \{n\}} \geq Z$ for some agent labeled n without loss of generality. Then the convexity of F implies

$$F(x_n + Z) - F(Z + L) \leq F(x_N) - F(x_{N \setminus \{n\}} + L)$$

As before we replace $F(x_n + Z)$ by this upper bound and rearrange (27) to the more demanding

$$\sum_{[n-1]} F(x_i + Z) \leq F(x_{N \setminus \{n\}} + L) + (\ell - 1)F(Z + L) + hF(Z + H)$$

which for the convex function $\widetilde{F}(y) = F(y + L)$ and $\widetilde{Z} = (\ell - 1)L + hH$ is exactly (27) at x_{-n} for the guarantee $g_{(\ell-1),h}$.

We are left with the case where $x_{N \setminus \{i\}} \leq Z$ for all i for which the different terms under F in (27) are ranked as follows:

$$x_N, Z + L \leq x_i + Z \leq Z + H$$

and the distribution $(\frac{1}{n}, \frac{\ell}{n}, \frac{h}{n})$ on the support $x, Z + L, Z + H$ is a mean-preserving spread of the uniform distribution on the n inputs $x_i + Z$. So $g_{\ell,h}$ meets (27).

If $h \geq 1$ the symmetric proof starts by assuming $x_{N \setminus \{n\}} \leq Z$ and using the convexity inequality

$$F(x_{N \setminus \{n\}} + H) - F(x_N) \leq F(Z + H) - F(x_n + Z)$$

to obtain a more demanding inequality that is in fact (27) for $g_{\ell,h-1}$ and the function $\widehat{F}(y) = F(y + H)$.

Step 2 The guarantee $g_{\ell,h}$ is tight. We fix x_i and compute

$$g_{\ell,h}(x_i) + \ell g_{\ell,h}(L) + h g_{\ell,h}(H) = F(x_i + \ell L + hH)$$

So the profile $(x_i, \overset{\ell}{L}, \overset{h}{H})$ is in the contact set of $g_{\ell,h}$ at x_i and by Lemma 3.5 we are done.

Statement ii) The derivative of the gap function is $\frac{dF}{dx}(nx_i) - \frac{dF}{dx}(x_i + Z)$ which changes from negative to positive at $\frac{1}{n-1}Z$.

Statement iii) The equality $g_{\ell,h}(x_i) = una(x_i)$ is rearranged as$_i$:

$$F(x_i + Z) = \frac{1}{n}F(nx_i) + \frac{\ell}{n}F(Z + L) + \frac{h}{n}F(Z + H))$$

This contradicts the strict convexity of F if ℓ, h are both positive. If ℓ or h is zero we are dealing with g_{inc} or g^{inc} with unanimous contact points at L and H respectively.

9.6 Lemma 6.1

Fix $\mathcal{W}$ defined by (19) and the equicontinuous functions w_k. For "only if" we assume that $\mathcal{W}$ is supermodular. Fix two agents i, j and a $(n-2)$-profile $x_{-ij} \in [L, H]^{[n] \setminus i,j}$. For any 4-tuple x_i, y_i, x_j, y_j such that $x_i > y_i$ and $x_j > y_j$ supermodularity means

$$\mathcal{W}(x_i, x_j; x_{-ij}) - \mathcal{W}(y_i, x_j; x_{-ij}) \geq \mathcal{W}(x_i, y_j; x_{-ij}) - \mathcal{W}(y_i, y_j; x_{-ij})$$

Suppose $L < y_i < x_i < H$ and pick an arbitrary rank $k, k \leq n-1$: we can choose x_{-ij}, x_j and y_j s. t. in the profiles on the RH x_i and y_i are of rank k, while after increasing y_j to x_j they are of rank $k+1$ in the profiles on the LH. Then the inequality above reads

$$w_{k+1}(x_i) - w_{k+1}(y_i) \geq w_k(x_i) - w_k(y_i)$$

As x_i, y_i can be chosen arbitrary close to each other, this proves (20) at any interior point of $[L, H]$ where w_k is differentiable (that is, a. e.).

For "if" we assume (20) and fix x_{-ij}. For any x_i, y_j s. t. x_i has rank k in $(x_i, y_j; x_{-ij})$ we have $\partial_i \mathcal{W}(x_i, y_j; x_{-ij}) = \frac{dw_k}{dx}(x_i)$ (a. e.): if y_j is below x_i and jumps up to x_j above x_i then by (20) $\partial_i \mathcal{W}(x_i, x_j; x_{-ij})$ also increases (weakly) to $\frac{dw_{k+1}}{dx}(x_i)$. If x_i is not isolated in the profile $(x_i, y_j; x_{-ij})$ the same argument applies to the left and right derivatives of $\mathcal{W}$ in x_i.

9.7 Theorem 6.1

We fix $\mathcal{W}$ given by (19) and supermodular, so $\frac{dw_k}{dx}(\cdot)$ increases weakly with k.

Step 1. *For any c the function g_c defined by (21) is in $\mathcal{G}^-$.* We saw in Definition 6.2 that it is enough to show $g_c \in \mathbf{G}^-$.

Because $g_c(x_i)$ and $\mathcal{W}(x_i; c)$ are continuous in x_i, c it is enough to prove the LH inequality (4) for strictly decreasing sequences $\{x_\ell\}_1^n$ and $\{c_k\}_1^{n-1}$ such that $H > c_1$ and $c_{n-1} > L$ and moreover $x_\ell \neq c_k$ for all ℓ, k. These assumptions hold for all the sequences x, c below.

Step 1.1 Call the profile of types x^* *regular* if

$$x_1^* > c_1 > x_2^* > c_2 > \cdots > c_{k-1} > x_k^* > c_k > \cdots > c_{n-1} > x_n^* \tag{29}$$

then compute

$$\sum_1^n g_c(x_k^*) = \sum_1^n \mathcal{W}(x_k^*, c) - \sum_1^{n-1} \mathcal{W}(c_k, c) = \sum_1^{n-1} (w_k(x_k^*) - w_k(c_k)) + \mathcal{W}(x_n^*, c) = \mathcal{W}(x^*)$$

so that x^* is a contact profile of g_c.

Step 1.2 For any three sequences x, x' and c we say that x' is reached from x by an *elementary jump up above* c_k if there is some ℓ such that $x_{-\ell} = x'_{-\ell}$; c_k is adjacent to x_ℓ in x from above and adjacent to x'_ℓ in x' from below. In other words: $x'_\ell > c_k > x_\ell$ and there is no other element of x or c between x_ℓ and x'_ℓ. The definition of an elementary jump down below c_k is exactly symmetrical.

We claim that for any sequence $\widetilde{x}$ we can find a regular profile x^* and a path (a sequence of sequences) $\sigma = \{\widetilde{x} = x^1, \cdots, x^t, \cdots, x^T = x^*\}$ such that
1) each step from x^t to x^{t+1} is an elementary jump up or down of some x_ℓ^t over some c_k
2) $\ell \leq k$ if x_ℓ^t jumps up above c_k, and $\ell \geq k+1$ if x_ℓ^t jumps down below c_k.

The proof by induction on n starts by distinguishing

Case 1: $\widetilde{x}_1 > c_1$. Then $\widetilde{x}_1$ never moves and $\widetilde{x}_1 = x_1^*$; if $\widetilde{x}_2, \cdots, \widetilde{x}_\ell$ are above c_1 then $\ell - 1$ successive elementary jumps down of these below c_1 defines the first $\ell - 1$ steps of the desired path; continuing until there are none, it remains to construct a path from the shorter sequence $\widetilde{x}_{-1}$ into a one regular w. r. t. the sequence c_{-1} by invoking the inductive assumption.

Case 2: $c_1 > \widetilde{x}_1$. Then the successive elementary jumps up of $\widetilde{x}_1$ over the closest c_k then $c_{k-1}, \cdots, c_1$ define the first k steps of the desired path until $x^{k+1} = x_1^*$ that never moves again; then we proceed with the shorter sequences $\widetilde{x}_{-1}$ and c_{-1} by the inductive assumption.

Step 1.3 We pick an arbitrary profile $\widetilde{x}$ and construct a sequence σ from $\widetilde{x}$ to some regular x^*, and check that in each step of the sequence the sum $\sum_1^n g_c(x_\ell) - \mathcal{W}(x)$ cannot decrease, which together with Step 1.1 concludes the proof that $g_c \in \mathbf{G}^-$. This sum develops as

$$\overbrace{(\sum_{\ell=1}^n \mathcal{W}(x_\ell, c))}^{B} - \overbrace{\mathcal{W}(x)}^{C} - \overbrace{\sum_{k=1}^{n-1} \mathcal{W}(c_k, c)}^{D}$$

Consider a jump up of x_ℓ^t above c_k: $x_\ell^{t+1} > c_k > x_\ell^t$. The net changes to the three terms in the sum are

$$\begin{aligned}\Delta B &= w_k(x_\ell^{t+1}) - w_{k+1}(x_\ell^t) + w_{k+1}(c_k) - w_k(c_k)\\ \Delta C &= w_\ell(x_\ell^{t+1}) - w_\ell(x_\ell^t)\ ;\ \Delta D = 0\end{aligned}$$

With the notation $\Delta f(a \to b) = f(b) - f(a)$ and some rearranging this gives

$$\Delta B - \Delta C + \Delta D = \Delta(w_k - w_\ell)(c_k \to x_\ell^{t+1}) + \Delta(w_{k+1} - w_\ell)(x_\ell^t \to c_k)$$

where both final Δ terms are non negative because $\ell \leq k$ and by (20) $w_k - w_\ell$ and $w_{k+1} - w_\ell$ increase weakly.

The proof for a jump down step is quite similar by computing the variation of $\sum_1^n g_c(x_\ell) - \mathcal{W}(x)$ to be $\Delta(w_\ell - w_k)(c_k \to x_\ell^t) + \Delta(w_\ell - w_{k+1})(x_\ell^{t+1} \to c_k)$ and recalling that in this case we have $\ell \geq k+1$.

Step 2 *A tight guarantee $g \in \mathcal{G}^-$ takes the form g_c in (21).*

Recall the notation $\mathcal{C}(g)$ for the set of contact profiles of g defined by (7). For each $k \in [n]$ its projection $\mathcal{C}_k(g)$ is the set of those $x_i \in [L, H]$ appearing in some profile $x \in \mathcal{C}(g)$ with the rank k; it is closed because $\mathcal{C}(g)$ is closed and we call its lower bound c_k. The sequence $\{c_k\}$ decreases weakly because in a contact profile where c_k is k-th the type x_{k+1} ranked $k+1$ is weakly below c_k. And $c_n = L$ because c_n is in some contact profile of g.

Check first that $\mathcal{C}_1(g) = [c_1, H]$ with the help of Lemma 3.9. For each $x_1 \in [c_1, H[$ where g is differentiable and x_1 appears with rank k in some contact profile we have $\frac{dg}{dx}(x_1) = \frac{dw_k}{dx}(x_1) \geq \frac{dw_1}{dx}(x_1)$ because $\mathcal{W}$ is supermodular. This implies $g(x_1) - g(c_1) \geq w_1(x_1) - w_1(c_1)$ everywhere in $[c_1, H]$.

Pick a profile $(c_1, x_{-1}) \in \mathcal{C}(g)$ where c_1 is ranked first and combine the latter inequality with this contact equation:

$$g(c_1) - w_1(c_1) = \sum_2^n (w_k(x_k) - g(x_k)) \leq g(x_1) - w_1(x_1)$$

The inequality above must be an equality because g is a lower guarantee therefore $\frac{dg}{dx}(x_1) = \frac{dw_1}{dx}(x_1)$ a.e. in $[c_1, H]$ and $[c_1, H] = \mathcal{C}_1(g)$.

We repeat this argument for $x_2 \in [c_2, c_1[$. In any of its contact profiles its rank is at least 2 by definition of c_1, so when g is differentiable at x_2 we have $\frac{dg}{dx}(x_2) = \frac{dw_k}{dx}(x_2) \leq \frac{dw_2}{dx}(x_2)$ by submodularity of $\mathcal{W}$. Then $g(x_2) \leq g(c_2) + w_2(x_2) - w_2(c_2)$ holds in $[c_2, c_1]$ and by plugging as above this inequality at a contact profile where c_2 is ranked second, we see that it is an equality and conclude that first, $\frac{dg}{dx}(x_2) = \frac{dw_2}{dx}(x_2)$ a.e. in $[c_2, c_1]$ and second, $[c_2, c_1] \subseteq \mathcal{C}_2(g)$.[15]

The clear induction argument gives $\frac{dg}{dx}(x_k) = \frac{dw_k}{dx}(x_k)$ a.e. in $[c_k, c_{k-1}]$; together with the continuity of g it implies that g is entirely determined by the value $g(L)$. But for $c = (c_1, \cdots, c_{n-1})$ the tight lower guarantee g_c ((21)) meets precisely the same differential system, therefore g and g_c differ by a constant; if they don't coincide g is either not a lower guarantee or not tight.

[15] Note that $\mathcal{C}_2(g)$ can extend beyond c_1 but this can only happen if $\frac{dw_2}{dx} = \frac{dw_1}{dx}$ in the overlap interval. To see this compare two contact profiles x and y such that $x^1 \geq x^2 > y^1 \geq y^2$ and use the LH of (4) at the two profiles where x^2 and y^2 have been swapped plus supermodularity of $\mathcal{W}$ to deduce that they are contact profiles as well.

9.8 Example 6.3

We can without loss assume that F is the identity because the change of variable $y_i = F(x_i)$ reaches that problem (Lemma 3.7). The proof resembles that of Proposition 2.1.

Fix a tight upper guarantee $g^+ \in \mathcal{G}_k^+$ and recall that g^+ is weakly increasing (Lemma 3.4). Define $p = ng^+(L)$: from $una(x_i) = \frac{1}{n}x_i$ and inequality (6) (Lemma 3.2) we get $p \geq L$. Observe next that $g_H(x_i) \equiv \frac{1}{n}H$ is in $\mathbf{G}_k^+$ (in fact also in $\mathcal{G}_k^+$ as we show below); if $p > H$ then g^+ is everywhere larger than g_H, a contradiction. So $p \in [L, H]$.

Apply now the feasibility inequality (4) to g^+ at the profile $(\overset{n-k}{L}, \overset{k}{x_i})$:

$$\frac{n-k}{n}p + kg^+(x_i) \geq x_i$$

If $k = n$ this gives $g^+(x_i) \geq una(x_i)$: as $una \in g^+$ we conclude $g^+ = una$. For $k \leq n-1$ we combine the inequality above with $g^+(x_i) \geq \frac{1}{n}p$ and obtain

$$g^+(x_i) \geq \max\{\frac{1}{n}p, \frac{1}{k}(x_i - \frac{n-k}{n}p)\} = \frac{1}{n}p + \frac{1}{k}(x_i - p)_+$$

It remains to check that the function on the right, which we write g_p^+, is itself an upper guarantee. Pick an arbitrary profile $x \in [L, H]^{[n]}$ and suppose that p is s. t. $x^\ell \geq p \geq x^{\ell+1}$. We must show

$$\sum_{[n]} g_p^+(x_i) = p + \frac{1}{k}((\sum_{t=1}^{\ell} x^t) - \ell p) \geq x^k$$

If $p \geq x^k$ we are done because the term in parenthesis is non negative. Assume now $p < x^k$ so that $x^k \geq \cdots \geq x^\ell \geq p \geq x^{\ell+1}$, then note that $(\sum_{t=1}^{\ell} x^t) - \ell p \geq k(x^k - p)$ and we are done.

The proof that for $k \geq 2$ the set $\mathcal{G}_k^-$ is also parametrised by $q \in [L, H]$ as

$$g_p^-(x_i) \geq \frac{1}{n}q + \frac{1}{n-k+1}(x_i - q)_-$$

and for $k = 1$ contains only una, is entirely similar.

9.9 Lemma 7.2

Statement i) is clear because $\mathcal{W}$ is symmetric. In *Statement ii)* upper-hemi-continuity of φ is clear because $\mathcal{W}$ and g are both continous (step 1 in the proof of Lemma 3.5 above).

To check that φ is convex valued we fix $(x_1, x_2), (x_1, x_2') \in \Gamma(\varphi)$ and z s. t. $x_2 < z < x_2'$, and check that $\Gamma(\varphi)$ contains (x_1, z) too. Pick some $w \in \varphi(z)$: if $w > x_1$ we see that $\Gamma(\varphi)$ contains (x_1, x_2) and (w, z) s.t. $(x_1, x_2) \ll (w, z)$ which is a contradiction by Lemma 7.1. If $w < x_1$ we use instead (w, z) and (x_1, x_2') to reach a similar contradiction, and we conclude $w = x_1$.

The proof below that φ is single-valued a. e. will complete that of statement *ii*).

Statement iii) If $x_1 < x_1'$ in $\mathcal{X}$ and $\varphi^-(x_1) < \varphi^+(x_1')$ we again contradict the strict super-modularity of $\mathcal{W}$ (Lemma 7.1) . So $x_1 < x_1' \implies \varphi^-(x_1) \geq \varphi^+(x_1')$ and φ^- and φ^+ are weakly decreasing.

If $\varphi(x_1)$ is not a singleton, $\varphi^+(x_1) > \varphi^-(x_1)$, then φ^+ jumps down at x_1; a weakly decreasing function can only do this a countable number of times. That the u.h.c. closure of φ^+ contains $[\varphi^-(x_1), \varphi^+(x_1)]$ follows from $\varphi^-(x_1) \geq \varphi^+(x_1 + \delta)$ for any $\delta > 0$.

Statement iv) If $\varphi(L)$ does not contain H we pick some x_1 in $\varphi(H)$: by statement *i)* $\varphi(x_1)$ contains H therefore $x_1 > L$; we reach a contradiction again from Lemma 7.1 because $\Gamma(\varphi)$ contains $(L, \varphi^+(L))$ and the strictly larger (x_1, H).

Statement v) Kakutani's theorem implies that at least one fixed point exists. If $\Gamma(\varphi)$ contains both (a, a) and (b, b) we contradicts again Lemma 7.1. Check finally that the inequalities $\varphi^-(a) < a < \varphi^+(a)$ are not compatible. Pick $\delta > 0$ s.t. $\varphi(a)$ contains $a - \delta$ and $a + \delta$: then $\Gamma(\varphi)$ contains $(a, a + \delta)$ and $(a - \delta, a)$ (by symmetry) and we invoke Lemma 7.1 again.

9.10 Theorem 7.1

Step 0: the integral in (25) is well defined.

For any correspondence φ as in Lemma 7.2 the integral $\int_a^{x_1} \partial_1 \mathcal{W}(t, \varphi(t))dt$ is the value of $\int_a^{x_1} \partial_1 \mathcal{W}(t, f(t))dt$ for any single-valued selection f of φ: this is independent of the choice of f because φ is multi-valued only at a countable number of points and every single-valued selection of $\varphi(x_1)$ is a measurable function.

Statement ii) Fix $g \in \mathcal{G}^-$ and its contact correspondence φ. The function $\mathcal{W}$ is uniformly Lipschitz in $[L, H]^2$ so by Lemma 3.8 g is Lipschitz as well, hence differentiable a. e.. The derivative $\frac{dg}{dx}$ is given by property (9) in Lemma 3.9: given x_1 for any $x_2 \in \varphi(x_1)$ we have $\frac{dg}{dx}(x_1) = \partial_1 \mathcal{W}(x_1, x_2)$, therefore we can write the RH as $\partial_1 \mathcal{W}(x_1, \varphi(x_1))$ without specifying a particular selection of $\varphi(x_1)$.

Note that $g(a) = una(a)$ because $(a, a) \in \Gamma(\varphi)$. Now integrating the differential equation above with this initial condition at a gives the desired representation (25).

Statement *i)*

Step 1 Lemma 7.2 implies that $\Gamma(\varphi)$ is a one-dimensional line connecting (L, H) and (H, L) that we can parametrise by a smooth mapping $s \to (\xi_1(s), \xi_2(s))$ from $[0, 1]$ into $[L, H]^2$ s.t. $\xi_1(\cdot)$ increases weakly from L to H and $\xi_2(\cdot)$ decreases weakly from H to L. We can also choose this mapping so that $\xi_1(\frac{1}{2}) = \xi_2(\frac{1}{2}) = a$, the fixed point of φ.[16]

We fix an arbitrary selection γ of φ, an arbitrary $\overline{x}_1$ in $[L, H]$, and check the identity

$$\int_a^{\overline{x}_1} \partial_1 \mathcal{W}(t, \varphi(t))dt + \int_a^{\gamma(\overline{x}_1)} \partial_1 \mathcal{W}(t, \varphi(t))dt = \mathcal{W}(\overline{x}_1, \gamma(\overline{x}_1)) - \mathcal{W}(a, a) \tag{30}$$

We change the variable t to s by $t = \xi_1(s)$ in the former and by $t = \xi_2(s)$ in the latter. Next $\overline{s}$ is the parameter at which $(\xi_1(\overline{s}), \xi_2(\overline{s})) = (\overline{x}_1, \gamma(\overline{x}_1))$ and we rewrite the LH above as

$$\int_{\frac{1}{2}}^{\overline{s}} \partial_1 \mathcal{W}(\xi_1(s), \xi_2(s)) \frac{\partial \xi_1}{\partial s}(s)ds + \int_{\frac{1}{2}}^{\overline{s}} \partial_1 \mathcal{W}(\xi_2(s), \xi_1(s)) \frac{\partial \xi_2}{\partial s}(s)ds$$

where in each term $\partial_1 \mathcal{W}(t, \varphi(t))$ we can select a proper selection of the (possible) interval because $(\xi_1(s), \xi_2(s)) \in \Gamma(\varphi)$. As $\mathcal{W}(x_1, x_2)$ is symmetric in x_1, x_2, we can replace the second integral by $\int_{\frac{1}{2}}^{\overline{s}} \partial_2 \mathcal{W}(\xi_1(s), \xi_2(s)) \frac{\partial \xi_2}{\partial s}(s)ds$ and conclude that the sum is precisely

$$\mathcal{W}(\xi_1(\overline{s}), \xi_2(\overline{s})) - \mathcal{W}(\xi_1(\frac{1}{2}), \xi_2(\frac{1}{2})) = \mathcal{W}(\overline{x}_1, \gamma(\overline{x}_1)) - \mathcal{W}(a, a)$$

Step 2 We show that (25) defines a bona fide guarantee g: $g(x_1) + g(x_2) \leq \mathcal{W}(x_1, x_2)$ for $x_1, x_2 \in [L, H]$.

[16] If a is 0,or 1 we check that (25) defines the two canonical incremental guarantees in Proposition 4.2.

The identity (30) amounts to $g(x_1) + g(\gamma(x_1)) = \mathcal{W}(x_1, \gamma(x_1))$ for all x_1. If we prove that $g \in \boldsymbol{G}^-$ this will imply it is tight. Compute

$$g(x_1) + g(x_2) = \mathcal{W}(x_1, \gamma(x_1)) + g(x_2) - g(\gamma(x_1)) = \mathcal{W}(x_1, \gamma(x_1)) + \int_{\gamma(x_1)}^{x_2} \partial_1 \mathcal{W}(t, \varphi(t))dt$$

We are left to show

$$\int_{\gamma(x_1)}^{x_2} \partial_1 \mathcal{W}(t, \varphi(t))dt \leq \mathcal{W}(x_1, x_2) - \mathcal{W}(x_1, \gamma(x_1)) \tag{31}$$

We assume without loss $x_1 \leq x_2$ and distinguish several cases by the relative positions of a and x_1, x_2 .

Case 1: $a \leq x_1 \leq x_2$, so that $\gamma(x_1) \leq a$. For every $t \geq \gamma(x_1)$ property *iii)* in Lemma 7.2 implies $\varphi^+(t) \leq \varphi^-(\gamma(x_1))$ and $\varphi(\gamma(x_1))$ contains x_1: therefore submodularity of $\mathcal{W}$ implies $\partial_1 \mathcal{W}(t, \varphi(t)) \leq \partial_1 \mathcal{W}(t, x_1)$ and

$$\int_{\gamma(x_1)}^{x_2} \partial_1 \mathcal{W}(t, \varphi(t))dt \leq \int_{\gamma(x_1)}^{x_2} \partial_1 \mathcal{W}(t, x_1)dt = \mathcal{W}(x_2, x_1) - \mathcal{W}(\gamma(x_1), x_1)$$

Case 2: $x_1 \leq a \leq \gamma(x_1) \leq x_2$. Similarly for $t \geq \gamma(x_1)$ we have $\varphi^+(t) \leq \varphi^-(\gamma(x_1))$ and conclude as in Case 1.

Case 3: $x_1 \leq x_2 \leq a$, so that $\gamma(x_1) \geq a$. For all $t \leq \gamma(x_1)$ we have $\varphi^-(t) \geq \varphi^+(\gamma(x_1))$ and $\varphi(\gamma(x_1))$ contains x_1: now submodularity of $\mathcal{W}$ gives $\partial_1 \mathcal{W}(t, z) \geq \partial_1 \mathcal{W}(t, x_2)$ for z between x_2 and $\gamma(x_1)$ and the desired inequality because the integral in (31) goes from high to low.

Case 4: $x_1 \leq a \leq x_2 \leq \gamma(x_1)$. Same argument as in Case 3.